

NON COMPLETELY SOLVABLE SYSTEMS OF COMPLEX FIRST ORDER PDE'S

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INTRODUCTION

History and motivation. Hans Lewy in [29] and Louis Nirenberg in [35] gave two fundamental results in the theory of linear partial differential equations. The first showed that a non homogeneous equation for a first order partial differential operator with complex valued real analytic coefficients, but C^∞ -smooth right hand side, may, in general, have no local weak solution. The second, that a homogeneous equation for a first order partial differential with complex valued smooth coefficients may have no non constant weak local solutions. Both results were formulated and proved for partial differential operators in \mathbb{R}^3 . A fuller understanding of [29] opened different directions of investigation (see e.g. [3, 4, 19, 36, 37]), especially from the two points of view of p.d.e. theory and of the analysis of CR manifolds.

Nirenberg's example was especially relevant to the problem of embedding CR manifold into complex manifolds. From this point of view, there have been two types of results. The Nirenberg example means that pseudoconvex three dimensional CR hypersurfaces cannot be locally CR -embedded. However the existence of sufficiently many independent solutions of the tangential Cauchy-Riemann equations was shown to hold for pseudoconvex higher dimensional CR hypersurfaces (see e.g. [1, 9, 12, 25, 26, 27]), and some general results were also obtained in terms of Lie algebras of vector fields (see e.g. [5, 17]). In the opposite direction, the counterexample of [35] was extended to CR hypersurfaces with degenerate or non degenerate Lorentzian signature (see e.g. [14, 15, 20, 22, 23]).

The results above were all obtained for the case of CR hypersurfaces. For higher codimension, a crucial invariant is the scalar Levi form, which is parametrized by the characteristic codirections of the tangential Cauchy-Riemann complex. The first result in higher codimension on the absence of the Poincaré lemma at the place q when some non degenerate scalar Levi form has q positive eigenvalues was first proved in [3]. In [18] this result was extended to some cases where the scalar Levi form is allowed to degenerate. Much less is known about the CR -embedding of manifolds of higher CR codimension. In [31] some results of [22] are extended under some supplementary conditions of the Cauchy-Riemann distribution. We also cite some partial results in [10, 11, 16].

Here we want to reconsider some of these questions, also in the more general framework of general distributions of complex vector fields of [2].

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Contents of the paper. Let M be a smooth paracompact manifold of dimension m , and let L_1, \dots, L_n be smooth complex vector fields on M . In local coordinates each L_j can be written as

$$(0.1) \quad L_j = L_j(x, D) = \sum_{i=1}^m a_{j,i}(x) \frac{\partial}{\partial x_i},$$

with coefficients $a_{j,i}$ which are assumed to be complex valued and C^∞ -smooth. We are interested in considering local solutions of the homogeneous system

$$(0.2) \quad L_j u = 0, \quad \text{for } j = 1, \dots, n.$$

When $n > 1$, since every local distribution solution u of (0.2) also satisfies

$$[L_{j_1}, L_{j_2}]u = (L_{j_1}L_{j_2} - L_{j_2}L_{j_1})u = 0, \dots, [L_{j_1}, [L_{j_2}, [\dots, L_{j_r}]]]u = 0$$

for all sequence j_1, j_2, \dots, j_r with $1 \leq j_1, j_2, \dots, j_r \leq n$, it is not restrictive to require that L_1, \dots, L_n satisfy the formal Cartan integrability conditions, i.e. that all commutators $[L_{j_1}, L_{j_2}]$ are linear combinations, with smooth coefficients, of L_1, \dots, L_n .

When this condition is satisfied, and L_1, \dots, L_n define linearly independent tangent vectors on a neighborhood U_0 of a point $p_0 \in M$, there are at most $m - n$ solutions u_1, \dots, u_{m-n} of (0.2), with $du_1(p_0), \dots, du_{m-n}(p_0)$ linearly independent. In fact this is always the case when the L_j 's have coefficients that are real analytic in some coordinate neighborhood of p_0 . Nirenberg's result in [35] shows that in general this is not true in the C^∞ case if $n = 1, m = 3$. A small perturbation of a vector field for which (0.2) has two analytically independent solutions changes to a vector field for which all local solutions of (0.2) are constant.

In §1 we show how this result extends to the case where $n = 1$, but m is allowed to be any integer larger or equal to 3. Namely, we show that the smooth complex vector fields for which (0.2) admits non locally constant solutions near some point of M form a small nowhere dense set of first Baire category in the Fréchet space of complex vector fields on M .

This generalization of [35] was already given in [31], and our main goal is to extend in fact the results of [22] to the case of higher CR codimension. We show that in general, given a smooth manifold M and any locally CR -embeddable Lorentzian CR structure on M , and a point $p_0 \in M$, there is a new Lorentzian CR structure, which is defined on a neighborhood of p_0 in M , and agrees to infinite order with the original one at p_0 , which is not locally CR -embeddable. We also show that the corresponding system (0.2) is not completely integrable in the class C^1 .

In §2 we collect the notions on CR manifolds that will be employed throughout the rest of the paper. In §3, §4, §5, §6 we prove the analog of the result of §1 for overdetermined systems by adaptations of the arguments therein. The results are weaker than those obtained for a scalar p.d.e. In fact, our constructions involve perturbations of an original system which, to keep formal integrability, employ either functions that are constant with respect to some variables, or, in §5, special morphisms of CR manifolds, and, in the more special cases of §6.7, analytic objects, called CR -divisors. In general, we obtain new overdetermined systems which are only defined in small coordinate neighborhoods. In §6.8 we prove that we can globally define a new CR structure on the Lorentzian real quadric Q in \mathbb{CP}^v which is not locally CR -embeddable at all points of a hyperplane section.

In §5 we also describe the CR complexes and show in §6.2 how the technique used in the rest of the paper can be also employed to give proofs of the non validity of the Poincaré lemma different from those of [3, 4, 18].

1. HOMOGENEOUS EQUATIONS WITH NO NONTRIVIAL SOLUTIONS

In this section we prove a generalization to dimensions ≥ 3 of a remarkable theorem of Nirenberg about local homogeneous solutions to a single homogeneous linear partial differential equation having smooth variable complex coefficients ([35], see also [22, 31]).

Here and in the following sections, M will denote a smooth paracompact real manifold of dimension m .

We denote by $\mathfrak{X}^{\mathbb{C}}(M)$ the Fréchet space of all C^∞ complex vector fields on M . Note that $\mathfrak{X}^{\mathbb{C}}(M)$ includes also real vector fields on M . When M is an open set in \mathbb{R}^m , each $L \in \mathfrak{X}^{\mathbb{C}}(M)$ can be written as

$$L = a_1(x) \frac{\partial}{\partial x_1} + a_2(x) \frac{\partial}{\partial x_2} + \cdots + a_m(x) \frac{\partial}{\partial x_m},$$

where $x = (x_1, x_2, \dots, x_m)$, and the coefficients $a_j(x) \in C^\infty(M)$ are (in general) complex valued.

Theorem 1.1. *Let M be a smooth manifold of dimension $m \geq 3$. Then the set \mathfrak{E} of $L \in \mathfrak{X}^{\mathbb{C}}(M)$ for which there exists a non empty open subset U of M , an $\epsilon > 0$, and a solution $u \in C^{1+\epsilon}(U)$ of $Lu = 0$ on U with $du(p) \neq 0$ for at least one $p \in U$, is a nowhere dense set of first (thin) Baire category.*

In other words: the set of all L on M having the property that any u with Hölder continuous first derivatives, which is a local solution to $Lu = 0$, in any neighborhood of any point, must be constant, is a dense set of the second (thick) Baire category.

First we prove a Lemma.

Lemma 1.2. *Let M be a smooth Riemannian manifold of dimension $m \geq 3$, and let $L_0 \in \mathfrak{X}^{\mathbb{C}}(M)$. Then for every point $p_0 \in M$, $h \in \mathbb{N}$, and $\epsilon > 0$ we can find $L \in \mathfrak{X}^{\mathbb{C}}(M)$ with*

$$(1.1) \quad \|L - L_0\|_{h,M} < \epsilon \quad \text{on } M,$$

such that

$$(1.2) \quad u \in C^1(U), \quad U^{\text{open}} \ni p_0, \quad Lu = 0 \text{ on } U \implies du(p_0) = 0.$$

Proof. We can argue on a small coordinate patch Ω about p_0 , and then, substituting L_0 by another vector field L_0 sufficiently close in the h -norm, we can assume that the coefficients of L_0 are real analytic in the coordinates in Ω , and that

$$L_0(p), \bar{L}_0(p), [L_0, \bar{L}_0](p) \text{ are linearly independent in } \mathbb{C}T_p M, \forall p \in \Omega.$$

Let $k = m - 2$. By the real analyticity assumption, using the Cauchy-Kowalevski theorem, and by shrinking Ω if needed, we can find $k + 1$ complex valued real analytic z_0, z_1, \dots, z_m on Ω with

$$L_0 z_i = 0, \text{ for } i = 0, \dots, k, \quad dz_0 \wedge d\bar{z}_0 \wedge dz_1 \wedge \cdots \wedge dz_k \neq 0 \quad \text{on } \Omega.$$

Let $x_i = \text{Re } z_i$, $y_i = \text{Im } z_i$. We can also arrange that $x_0, y_0, x_1, \dots, x_k$ are real coordinates in Ω centered at p_0 , and that $y_i(p_0) = 0$, $dy_i(p_0) = 0$ for $i = 1, \dots, k$.

This preparation yields a local CR -embedding of Ω as a CR submanifold of CR dimension 1 and CR codimension k in \mathbb{C}^{k+1} , given by

$$y_i = h_i(z_0, x) \text{ for } i = 1, \dots, k,$$

$$\text{with } x = (x_1, \dots, x_k), \text{ and } h_i = O(z_0 \bar{z}_0 + |x|^2).$$

After multiplication by a nowhere zero function, we can take L_0 of the form

$$L_0 = \frac{\partial}{\partial \bar{z}_0} + \sum_{i=1}^k a_i \frac{\partial}{\partial x_i}, \quad a_i \in C^\infty(\Omega).$$

The condition that $[L_0, \bar{L}_0](p_0) \neq 0$ implies that $\partial^2 h_i / \partial z_0 \partial \bar{z}_0 \neq 0$ at p_0 for some index i . Moreover, we note that we obtain new solutions of the homogeneous equation $L_0 u = 0$ by taking for u any holomorphic function of z_0, z_1, \dots, z_k . This allows us to use biholomorphic transformations to obtain that

(*) the real Hessian of $h_1(z_0, x)$ is positive definite in Ω .

In this way, the sets $\Omega_r = \{p \in \Omega \mid \text{Im } z_1 < r\}$, for $r > 0$, form a fundamental system of open neighborhoods of p_0 in M . Set

$$M_\tau = \{p \in \Omega \mid z_1 = \tau\}, \text{ for } \tau \in \mathbb{C}.$$

Then, by (*), $M_0 = \{p_0\}$, and there is an open connected neighborhood ω of 0 in \mathbb{C} and a smooth real curve $\text{Im } \tau = \phi(\text{Re } \tau)$ in ω , passing through 0, with the properties

- (i) $M_\tau \subset \Omega$ if $\tau \in \omega$,
- (ii) $M_\tau = \emptyset$ if $\text{Im } \tau < \phi(\text{Re } \tau)$,
- (iii) $M_\tau = \{\text{a point}\}$ if $\text{Im } \tau = \phi(\text{Re } \tau)$,
- (iv) $M_\tau \simeq S^k$ if $\text{Im } \tau > \phi(\text{Re } \tau)$.

Let $\{D_\nu\}$ be a sequence of pairwise disjoint closed discs in

$$\omega^+ = \{\tau \in \omega \mid \text{Im } \tau > \phi(\text{Re } \tau)\},$$

with centers and radii converging to 0 for $\nu \rightarrow \infty$. For a suitable $r_0 > 0$, all sets

$$\omega_r^+ = \{\tau \in \omega \mid \phi(\text{Re } \tau) < \text{Im } \tau < r\}$$

are connected, for $0 < r < r_0$. Set

$$\omega_r' = \omega_r^+ \setminus \bigcup_\nu D_\nu, \quad \Omega_r' = \{p \in \Omega \mid z_1(p) \in \omega_r'\}.$$

Let u be a C^1 solution of $L_0(u) = 0$ on Ω_r' , for some $0 < r < r_0$. For each $\tau \in \omega_r'$ we define

$$F(\tau) = \int_{M_\tau} u dz_0 \wedge dz_2 \wedge \dots \wedge dz_k.$$

We claim that F is holomorphic in ω_r' . Let indeed κ be an arbitrary smooth simple closed curve in ω_r' . Then $\bigcup_{\tau \in \kappa} M_\tau$ is the boundary of a domain N_κ in Ω , that is diffeomorphic to the Cartesian product of a 2-disc and a $(k-1)$ -ball, and

$$\begin{aligned} \oint_\kappa F(\tau) d\tau &= \oint_\kappa d\tau \int_{M_\tau} u dz_0 \wedge dz_2 \wedge \dots \wedge dz_k = \pm \int_{\partial N_\kappa} u dz_0 \wedge dz_1 \wedge dz_2 \wedge \dots \wedge dz_k \\ &= \pm \int_{N_\kappa} du \wedge dz_0 \wedge dz_1 \wedge \dots \wedge dz_k = 0, \end{aligned}$$

because (see [28])

$$du \wedge dz_0 \wedge dz_1 \wedge \dots \wedge dz_k = (L_0 u) d\bar{z}_0 \wedge dz_0 \wedge dz_1 \wedge \dots \wedge dz_k = 0.$$

By Morera's theorem, F is holomorphic on ω'_r . Moreover, F extends to a continuous function on the closure of ω'_r in $\omega \cap \{\text{Im } \tau < r\}$, that equals 0 for $\text{Im } \tau = \phi(\text{Re } \tau)$ because of (iii). It follows that $F(\tau) = 0$ for $\tau \in \omega'_r$.

For each $\nu \in \mathbb{N}$, let $Q_\nu = \{p \in \Omega \mid z_1(p) \in D_\nu\}$. We fix smooth functions ψ_i in Ω , such that $\psi_i dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge dz_k$ is, for each $i = 0, 1, \dots, k$, a non-negative real regular measure, with

$$\text{supp } \psi_i = \bigcup_{j=0}^{\infty} Q_{i+j(k+1)}$$

and such that, for

$$L = L_0 + \psi_0 \frac{\partial}{\partial z_0} + \sum_{i=1}^k \psi_i \frac{\partial}{\partial x_i}$$

we have $\|L - L_0\|_h < \epsilon$. Assume now that $u \in C^1(\Omega_r)$ satisfies $Lu = 0$. Hence, for all ν sufficiently large, $Q_\nu \subset \Omega_r$, and

$$\begin{aligned} 0 &= \pm \oint_{\partial D_\nu} d\tau \int_{M_\tau} u dz_0 \wedge dz_2 \wedge \cdots \wedge dz_k = \int_{\partial Q_\nu} u dz_0 \wedge dz_1 \wedge \cdots \wedge dz_k \\ &= \int_{Q_\nu} (L_0 u) dz_0 \wedge d\bar{z}_0 \wedge dz_1 \wedge \cdots \wedge dz_k = \int_{Q_\nu} ((L_0 - L)u) dz_0 \wedge d\bar{z}_0 \wedge dz_1 \wedge \cdots \wedge dz_k \end{aligned}$$

implies, by the mean value theorem, that, for all large $i \in \mathbb{N}$, there are points $p_i, p'_i \in Q_j$ such that, for large j ,

$$\begin{cases} \text{Re } \frac{\partial u(p_{j(1+k)})}{\partial z_0} = 0, & \text{Im } \frac{\partial u(p'_{j(1+k)})}{\partial z_0} = 0, \\ \text{Re } \frac{\partial u(p_{i+j(k+1)})}{\partial x_i} = 0, & \text{Im } \frac{\partial u(p'_{i+j(k+1)})}{\partial x_i} = 0, \text{ for } i = 1, \dots, k. \end{cases}$$

By passing to the limit, as $p_j \rightarrow p_0$, we obtain that

$$\frac{\partial u(p_0)}{\partial z_0} = 0, \quad \frac{\partial u(p_0)}{\partial x_i} = 0, \text{ for } i = 1, \dots, k.$$

Together with $Lu(p_0) = 0$, this yields $du(p_0) = 0$. \square

Proof of Theorem 1.1. We fix a Riemannian metric on M , so that we can compute the length of vectors and covectors and the C^h -norms of functions defined on subsets of M . Then we have seminorms which endow $\mathfrak{X}(M)$ with a Fréchet space topology, and we may discuss Baire category. Let $\{U_\nu\}_{\nu \in \mathbb{N}}$ be a countable basis of non empty open subsets of M , and for each $\nu \in \mathbb{N}$ fix a point $p_\nu \in U_\nu$. For $h \in \mathbb{N}$ we define $\mathfrak{E}(\nu, h)$ to be the closure in $\mathfrak{X}^{\mathbb{C}}(M)$ of the set of L such that

$$(1.3) \quad \exists u \in C^{1+\frac{1}{h}}(U_\nu) \text{ with } \begin{cases} Lu = 0 & \text{on } U_\nu, \\ \|u\|_{1+\frac{1}{h}, U_\nu} \leq h, \\ |du(p_\nu)| \geq \frac{1}{h}. \end{cases}$$

The set $\mathfrak{E}(\nu, h)$ has an empty interior. This can be proved by contradiction. If some $\mathfrak{E}(\nu, h)$ had an interior point, by Lemma 1.2 it would contain an interior point L satisfying (1.2) with $p_0 = p_\nu$. By definition, there is a sequence $\{L_j\}_{j \in \mathbb{N}}$ with $L_j \rightarrow L$ in $\mathfrak{X}^{\mathbb{C}}(M)$ for $j \rightarrow \infty$ such that for each j , there is $u_j \in C^{1+\frac{1}{h}}(U_\nu)$ with $L_j u_j = 0$ on U_ν , $\|u_j\|_{1+\frac{1}{h}, U_\nu} \leq h$ and $|du(p_\nu)| \geq \frac{1}{h}$. By the Ascoli-Arzelà theorem, passing to a subsequence we can assume that $u_j \rightarrow u \in C^{1+\frac{1}{h}}(U_\nu)$, uniformly with

their first derivatives on every compact neighborhood of p_v in U_v . Then $Lu = 0$ on U_v , and $|du(p_v)| \geq \frac{1}{h} > 0$ contradicts (1.2).

Therefore the union $\bigcup_{v,h} \mathfrak{E}(v, h)$ is a countable union of closed subsets having empty interior, hence of first Baire category. Then also \mathfrak{E} is of first Baire category, because $\mathfrak{E} \subset \bigcup_{v,h} \mathfrak{E}(v, h)$. This completes the proof of the Theorem. \square

2. INVOLUTIVE SYSTEMS AND CR MANIFOLDS

To extend the result of §1 to overdetermined systems of homogeneous first order p.d.e.'s, we will develop ideas from [14, 15, 20]. In this section we begin by describing the general framework. In the following, M will denote a C^∞ -smooth manifold of real dimension m .

2.1. Generalized complex distributions and CR structures. Let \mathcal{Z} be a generalized distribution of smooth complex vector fields on M . This means that \mathcal{Z} defines, for each open subset U of M a, $C^\infty(U)$ submodule of $\mathfrak{X}^\mathbb{C}(U)$, in such a way that the assignment $U \rightarrow \mathcal{Z}(U)$ is a sheaf:

- (1) If $U^{\text{open}} \subset V^{\text{open}} \subset M$, then $Z|_U \in \mathcal{Z}(U)$ for all $Z \in \mathcal{Z}(V)$;
- (2) If $\{U_v\}$ is a family of open subsets of M , a smooth complex vector field Z , defined on $\bigcup_v U_v$, belongs to $\mathcal{Z}(\bigcup_v U_v)$ if and only if $Z|_{U_v} \in \mathcal{Z}(U_v)$ for all v .

Our main interest in the sequel will be focused on the *local* solutions to the homogeneous system

$$(2.1) \quad Zu = 0, \quad \forall Z \in \mathcal{Z}.$$

It is therefore natural to assume in the following that \mathcal{Z} is *involutive*, or *formally integrable*. This means that

$$(2.2) \quad [Z_1, Z_2] \in \mathcal{Z}(U), \quad \forall Z_1, Z_2 \in \mathcal{Z}(U), \quad \forall U^{\text{open}} \subset M.$$

Since \mathcal{Z} is a fine sheaf, every germ $Z_{(p)}$ of \mathcal{Z} at a point $p \in M$ is the restriction of a global $Z \in \mathcal{Z}(M)$. Thus we can for simplicity utilize global sections $Z \in \mathcal{Z}(M)$ in most of the discussion below.

For each point $p \in M$, we consider the set

$$(2.3) \quad \mathbf{Z}_p M = \{Z(p) \mid Z \in \mathcal{Z}(M)\} \subset \mathbb{C}T_p M.$$

If the dimension of the \mathbb{C} -linear space $\mathbf{Z}_p M$ is constant, we say that \mathcal{Z} is a *distribution* of complex vector fields.

Definition 2.1. A *CR structure* on M is the datum of an involutive distribution \mathcal{Z} of smooth complex vector fields with $\mathcal{Z} \cap \overline{\mathcal{Z}} = 0$.

The constant dimension n of $\mathbf{Z}_p M$ is its *CR dimension*, and $k = m - 2n$ its *CR codimension*. We call the pair (n, k) the *type* of the CR manifold M .

In the case where \mathcal{Z} is a CR structure on M , we write sometimes $T^{0,1}M$ for $\mathbf{Z}M$.

When M is a real smooth submanifold of a complex manifold \mathbb{X} , we consider on M the generalized distribution

$$\mathcal{Z}(U) = \{Z \in \mathfrak{X}^\mathbb{C}(U) \mid Z_p \in T_p^{0,1}\mathbb{X}, \quad \forall p \in U\},$$

where $T^{0,1}\mathbb{X}$ is the bundle of anti-holomorphic complex tangent vectors to \mathbb{X} . Then \mathcal{Z} is involutive and $\mathcal{Z} \cap \overline{\mathcal{Z}} = 0$. When \mathcal{Z} has constant rank, \mathcal{Z} defines a CR structure on M , for which we say that M is a *CR-submanifold* of \mathbb{X} .

Let $1 \leq a \leq \infty$. A *complex CR-immersion* of class C^a of M is a C^a -smooth immersion $\phi : M \rightarrow \mathbb{X}$ of M into a complex manifold \mathbb{X} with $d\phi(\mathbf{Z}_p M) \subset T_{\phi(p)}^{0,1} \mathbb{X}$ for all $p \in U$.

For any open set U of M we set

$$(2.4) \quad \mathcal{O}_M(U) = \{u \in C^1(U) \mid Zu = 0, \forall Z \in \mathcal{Z}(M)\}.$$

The assignment $U^{open} \rightarrow \mathcal{O}_M(U)$ defines a sheaf of rings of germs of complex valued differentiable functions on M .

2.2. The differential ideal and complete integrability. Let $\Omega_M^* = \bigoplus_{0 \leq p \leq m} \Omega_M^p$ be the sheaf of germs of alternated smooth differential forms on M . We associate to \mathcal{Z} the *differential ideal*

$$(2.5) \quad \mathcal{J}_M = \bigoplus_{p \geq 1} \mathcal{J}_M^p \quad \text{with} \quad \mathcal{J}_M^p = \{\eta \in \Omega_M^p \mid \eta|_{\mathcal{Z}(M)} = 0\}.$$

This is a graded ideal sheaf of Ω_M^* , generated by its elements of degree 1. Being interested in the local solutions to (2.1), we can assume that \mathcal{J}_M is *complete* and that \mathcal{Z} is the *characteristic system* of \mathcal{J}_M , i.e. that

$$\begin{aligned} \mathcal{Z}(U) &= \{Z \in \mathfrak{X}^{\mathbb{C}}(M) \mid Z \lrcorner \mathcal{J}_M(U) \subset \mathcal{J}_M(U)\} \\ &= \{Z \in \mathfrak{X}^{\mathbb{C}}(U) \mid \eta(Z) = 0, \forall \eta \in \mathcal{J}_M^1(U)\}, \quad \forall U^{open} \subset M. \end{aligned}$$

If \mathcal{Z} is a distribution, it is the characteristic system of its differential ideal. The pointwise evaluation of the elements of \mathcal{J}_M^1 yields in this case a smooth subbundle $\mathbf{Z}^0 M$ of $\mathbb{C}T^*M$, given by

$$(2.6) \quad \mathbf{Z}^0 M = \bigsqcup_{p \in M} \mathbf{Z}_p^0 M, \quad \text{with} \quad \mathbf{Z}_p^0 M = \{\eta \in \mathbb{C}T_p^* M \mid \eta(Z) = 0 \forall Z \in \mathcal{Z}(M)\}.$$

In general, (2.6) defines a subset of the complexified tangent bundle of M .

Definition 2.2. Let \mathcal{Z} be a generalized distribution of smooth complex vector fields on M and $p_0 \in M$. We say that \mathcal{Z} is *completely integrable* at p_0 if

$$(2.7) \quad \forall \eta \in \mathbf{Z}_{p_0}^0 M \quad \exists u \in \mathcal{O}_{M(p_0)} \quad \text{with} \quad du(p_0) = \eta.$$

This means that (2.1) has at p_0 the largest number of differentially independent local solutions that is permitted by the rank of \mathcal{Z} .

2.3. The case of CR manifolds. Let \mathcal{Z} be a CR structure of type (n, k) on M . Complete integrability at $p_0 \in M$ is equivalent to the existence of a complex CR-immersion of class C^1 of an open neighborhood U of p_0 into \mathbb{C}^{n+k} .

The question of the regularity of complex CR-immersions seems in general a rather delicate open problem (see e.g. [30]). Note that any C^1 -immersion is in fact C^∞ -smooth when M satisfies suitable pseudo-concavity assumptions (see [2]).

For C^∞ -smooth complex local CR-immersions we introduce a special notation.

Definition 2.3. A CR-chart on M is the datum of an open subset U and of $n + k$ smooth CR functions $z_1, \dots, z_{n+k} \in \mathcal{O}_M(U) \cap C^\infty(U)$, such that

$$dz_1(p) \wedge \dots \wedge dz_{n+k}(p) \neq 0, \quad \forall p \in U.$$

Clearly $\phi(p) = (z_1(p), \dots, z_{n+k}(p))$ provides in this case a C^∞ -smooth CR-immersion of U in \mathbb{C}^{n+k} .

The functions z_1, \dots, z_{n+k} of a CR -chart are not independent complex coordinates when $k > 0$. For each point p_0 of U there are indeed k real valued functions ρ_1, \dots, ρ_k , defined and C^∞ on an open neighborhood G of $\phi(p_0)$ in \mathbb{C}^{n+k} , with $\rho_i(z_1, \dots, z_{n+k}) = 0$ on a neighborhood of p_0 , and $\partial\rho_1(\phi(p_0)) \wedge \dots \wedge \partial\rho_k(\phi(p_0)) \neq 0$.

Definition 2.4. We say that a CR manifold M is *locally CR -embeddable* if the open subsets U of its CR -charts make a covering.

Locally CR -embeddable CR manifolds can be abstractly defined as ringed spaces, using the structure sheaf $\mathcal{O}_M^\infty = \mathcal{O}_M \cap \mathcal{C}^\infty$ of the germs of its smooth CR functions.

Lemma 2.5. *Let M be a CR manifold of type (n, k) and $p_0 \in M$. Then we can find an open neighborhood U of p_0 in M and a new CR structure on M which is locally CR -embeddable and agrees to infinite order with the original one at p_0 .*

Proof. Let \mathcal{Z} be the CR structure on M . It suffices to consider smooth functions z_1, \dots, z_v which are defined on a neighborhood of p_0 , satisfy $Zz_j = 0^\infty$ at p_0 , and have $dz_1(p_0) \wedge \dots \wedge dz_v(p_0) \neq 0$. To prove the existence of such functions, we observe that it is always possible to find a smooth coordinate chart (U, x_1, \dots, x_m) centered at p_0 such that \mathcal{Z} is generated in U by vector fields of the form

$$Z_i = \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial x_{i+n}} + \sum_{j=n+1}^m a_j(x) \frac{\partial}{\partial x_j}, \quad \text{with } a_j(x) = O(|x|).$$

$$\text{Let } L_i = \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial x_{i+n}}, \text{ and } R_i = \sum_{j=n+1}^m a_j(x) \frac{\partial}{\partial x_j}.$$

We denote by \mathfrak{m} the maximal ideal of the local ring $\mathbb{C}\{\{x_1, \dots, x_m\}\}$ of formal power series of x_1, \dots, x_m . We obtain formal power series solution to (2.1) by constructing by recurrence sequences $\{f_h\}_{h \geq 0} \subset \mathbb{C}\{\{x_1, \dots, x_m\}\}$ which solve the equations

$$(*) \quad \begin{cases} f_h \in \mathfrak{m}^h, \\ L_j f_1 \in \mathfrak{m}, & \text{for } j = 1, \dots, n, \\ L_j f_{h+1} + R_j f_h \in \mathfrak{m}^{h+1}, & \text{for } j = 1, \dots, n. \end{cases}$$

We observe that, taking f_1 equal to $x_i + ix_{i+n}$ for $i = 1, \dots, n$, or to x_{2n+i} , for $i = 1, \dots, k$, we obtain v independent solutions of $L_i f_1 = 0$ for $1 \leq i \leq n$.

Assume now that $d \geq 1$ and $f_d \in \mathfrak{m}^d$ satisfies

$$L_i f_d + R_i f_{d-1} \in \mathfrak{m}^d, \quad \text{for } 1 \leq i \leq n.$$

The integrability conditions yield $[Z_i, Z_j] = 0$ for $1 \leq i, j \leq n$. Hence we obtain

$$(**) \quad 0 = [Z_i, Z_j] f_d = -L_i R_j f_d + L_j R_i f_d + [R_i, R_j] f_d.$$

We have $R_i f_d \in \mathfrak{m}^d$, and hence there is a polynomial $g_{i,d} \in \mathbb{C}[x_1, \dots, x_m]$, homogeneous of degree d , such that $R_i f_d - g_{i,d} \in \mathfrak{m}^{d+1}$. Since $[R_i, R_j] f_d \in \mathfrak{m}^{d+1}$, we obtain from (**) that $L_i g_{j,d} = L_j g_{i,d}$ for all $1 \leq i, j \leq n$ and therefore there is a polynomial $f_{d+1} \in \mathbb{C}[x_1, \dots, x_m]$, homogeneous of degree $d+1$, such that $L_i f_{d+1} = g_{i,d}$ for $i = 1, \dots, n$. The series $\sum f_d$ of the terms of a sequence $\{f_d\}$ solving (*) is a formal power series solution of (2.1).

In particular, we can find solutions $\{z_1\}, \dots, \{z_v\} \in \mathbb{C}\{\{x_1, \dots, x_m\}\}$ to (2.1) with $d\{z_i\}(0) = dx_i(0) + idx_{i+n}(0)$ for $i = 1, \dots, n$ and $d\{z_i\}(0) = dx_{n+i}(0)$ for $i = n+1, \dots, v$. It suffices then to take smooth functions z_1, \dots, z_v having Taylor series $\{z_1\}, \dots, \{z_v\}$ at 0. \square

2.4. Characteristic bundle and Levi form. [32] The underlying real distribution and the characteristic bundle of \mathcal{Z} are:

$$(2.8) \quad \mathcal{H} = \{\operatorname{Re} Z \mid Z \in \mathcal{Z}\}, \quad \text{i.e.} \quad \mathcal{H}(U) = \{\operatorname{Re} Z \mid Z \in \mathcal{Z}(U)\}, \quad \forall U^{\text{open}} \subset M,$$

$$(2.9) \quad H^0 M = \{\xi \in T^* M \mid \xi(X) = 0, \forall X \in \mathcal{H}(M)\}.$$

To each characteristic covector $\xi_0 \in H_{p_0}^0 M$ we associate a Hermitian symmetric form on $\mathbf{Z}_{p_0} M$, by

$$(2.10) \quad \mathfrak{L}_{\xi_0}(Z_1, Z_2) = i\xi_0([Z_1, \bar{Z}_2]), \quad \forall Z_1, Z_2 \in \mathcal{Z}(M).$$

In fact a straightforward verification shows that the value of the right hand side of (2.10) only depends on $Z_1(p_0), Z_2(p_0) \in \mathbf{Z}_{p_0} M$.

Moreover, $\mathfrak{L}_{\xi_0}(Z_1, Z_2) = 0$ if one of the two vector fields is real valued on a neighborhood of p_0 . Thus \mathfrak{L}_{ξ_0} defines a Hermitian symmetric form on the quotient of $\mathbf{Z}_{p_0} M$ by the subspace $\mathbf{N}_{p_0} M = \{Z(p_0) \mid Z \in \mathcal{Z}(M) \cap \overline{\mathcal{Z}(M)}\}$, consisting of the values at p_0 of the complex multiples of the real vector fields in $\mathcal{Z}(M)$. Set

$$(2.11) \quad \check{\mathbf{Z}}_{p_0} M = \mathbf{Z}_{p_0} M / \mathbf{N}_{p_0} M.$$

If $\xi_0 \in H_{p_0}^0 M$, then (2.10) defines a Hermitian symmetric form \mathbf{L}_{ξ_0} on $\check{\mathbf{Z}}_{p_0} M$, that we call the *Levi form* of \mathcal{Z} at ξ_0 .

Definition 2.6. Let $p_0 \in M$ and $\xi_0 \in H_{p_0}^0 M$. We say that \mathcal{Z} is q -pseudoconvex at ξ_0 if \mathbf{L}_{ξ_0} is nondegenerate and has exactly q positive eigenvalues on $\check{\mathbf{Z}}_{p_0} M$.

If \mathcal{Z} is 1-pseudoconvex at some $\xi_0 \in H_{p_0}^0 M$, we say that \mathcal{Z} is *Lorentzian* at p_0 .

If $\mathcal{Z}(M)$ is generated by a single vector field L near p_0 , the condition of being Lorentzian at p_0 means that $L(p_0)$, $\bar{L}(p_0)$, and $[L, \bar{L}](p_0)$ are linearly independent in $\mathbb{C}T_{p_0} M$.

2.5. Reduction of complete integrability to the case of CR manifolds. When $\mathbf{N}_p M$ has constant dimension on a neighborhood U of $p_0 \in M$, then the real vector fields in $\mathcal{Z}(U)$ define an involutive distribution \mathcal{N} of *real* vector fields on U . By the Frobenius theorem, there is an open neighborhood W of p_0 in U and a smooth fibration $\pi : W \rightarrow B$ of W such that B is a smooth manifold and the fibers of π are integral submanifolds of \mathcal{N} . One easily proves

Lemma 2.7. *There is a CR structure \mathcal{Z}' on B such that for every $p \in W$ we have $\mathcal{O}_{M,(p)} = \pi^* \mathcal{O}_{B,(\pi(p))}$, and \mathcal{Z} is completely integrable at $p \in W$ if and only if \mathcal{Z}' is completely integrable at $\pi(p)$.*

3. INVOLUTIVE SYSTEMS WHICH ARE NOT COMPLETELY INTEGRABLE AT p_0

In this section, we give a weak generalization of the results of §1 to Lorentzian CR manifolds M with arbitrary CR-codimension $k \geq 1$ and CR-dimension $n \geq 2$. We recall that $m = \dim_{\mathbb{R}} M = 2n + k$, and we set $v = n + k$.

We closely follow the arguments of §1.

Assume that M is locally CR-embeddable and Lorentzian at p_0 . Then there is a CR-chart (U, z_1, \dots, z_v) centered at p_0 , with $dz^i(p_0)$ real for $i = n + 1, \dots, v$, and

$$(3.1) \quad \operatorname{Im} z_v + z_v \bar{z}_v + \sum_{i=1}^{n-1} z_i \bar{z}_i = \sum_{i=n}^{v-1} z_i \bar{z}_i + O(|z|^3) \quad \text{on } U.$$

By shrinking, we get $\sum_{i=n}^{v-1} z_i \bar{z}_i \geq \frac{1}{2} \operatorname{Im} z_v$ on U .

We consider the map $\pi : U \ni p \rightarrow w = (z_1(p), \dots, z_{n-1}(p), z_v(p)) \in \mathbb{C}^n$. By a further shrinking, we can assume that there is an open ball $B \subset \mathbb{C}^n$, centered at 0, such that

- $\pi(U) = \bar{\omega}$, with $B \setminus \omega$ strictly convex, and $\partial\omega \cap B$ smooth;
- if $\text{Im } \tau \geq 0$, then $\{w \in B \mid \text{Im } w_n = \tau\} \subset \omega$;
- for all $w \in \omega$ the set $M_w = \pi^{-1}(w)$ is diffeomorphic to the sphere S^k ;
- for $w \in \partial\omega \cap B$ the set $M_w = \pi^{-1}(w)$ is a point.

As in §1, we have:

Lemma 3.1. *If $u \in \mathcal{O}_M(U)$, then*

$$(3.2) \quad F(w) = \int_{M_w} u dz_n \wedge \dots \wedge dz_{v-1} = 0, \quad \forall w \in \omega.$$

Proof. We prove first that F is holomorphic on ω .

Fix any polycylinder $D = D_1 \times \dots \times D_n$ in ω , with $D_i = \{\tau \in \mathbb{C} \mid |\tau - \tau_i| \leq \epsilon_i\}$. For $1 \leq j \leq n$ we set $\partial_j(D) = \{w \in D \mid |w_j - \tau_j| = \epsilon_j\}$, $\gamma_j = \frac{\partial}{\partial \bar{w}_j} \rfloor (d\bar{w}_1 \wedge \dots \wedge d\bar{w}_n)$

and consider the integral

$$\oint_{\partial_j D} F(w) dw_1 \wedge \dots \wedge dw_n \wedge \gamma_j = \oint_{\partial_j D} dw_1 \wedge \dots \wedge dw_n \wedge \gamma_j \int_{M_w} u dz_n \wedge \dots \wedge dz_{v-1}.$$

Let $N_i = \pi^{-1}(\partial_i D)$ and $N = \pi^{-1}(D)$. We have $\partial N = \sum_{i=1}^n \pm N_i$. Moreover, the form $u dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \gamma_j$ is zero on N_i for $i \neq j$. Thus we obtain:

$$\begin{aligned} \oint_{\partial_j D} F(w) dw_1 \wedge \dots \wedge dw_n \wedge \gamma_j &= \pm \int_{N_j} u dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \gamma_j \\ &= \sum_{i=1}^n \int_{N_i} \pm u dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \gamma_j = \pm \int_N du \wedge dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \gamma_j = 0 \end{aligned}$$

because $du \in \langle dz_1, \dots, dz_v \rangle$ by the assumption that $u \in \mathcal{O}_M(U)$. This equality, valid for all closed polycylinder D in ω and all $1 \leq j \leq n$, implies that F is holomorphic in ω . Clearly $F(w) \rightarrow 0$ when $w \rightarrow \partial\omega \cap B$, because M_{w_0} is a point for $w_0 \in \partial\omega \cap B$, and hence $F = 0$ on ω by Holmgren's uniqueness theorem, since $\bar{\partial}$ has constant coefficients in \mathbb{C}^n . \square

Let ψ be a smooth function with compact support in \mathbb{C} , and set

$$\hat{\psi}(\tau) = \frac{1}{2\pi i} \iint \frac{\psi(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - \tau}.$$

Then $\frac{\partial \hat{\psi}}{\partial \bar{\tau}} = \psi$ and therefore

$$\psi^\sharp(\zeta, \tau) = \bar{\zeta} \psi(\tau) d\bar{\tau} + \hat{\psi}(\tau) d\bar{\zeta} = \bar{\partial}(\bar{\zeta} \hat{\psi}(\tau))$$

is a $\bar{\partial}$ -closed form in \mathbb{C}^2 , with

$$d\psi^\sharp = \frac{\partial \hat{\psi}(\tau)}{\partial \tau} d\tau \wedge d\bar{\zeta} + \bar{\zeta} \frac{\partial \psi(\tau)}{\partial \tau} d\tau \wedge d\bar{\tau} = d\tau \wedge \frac{\partial}{\partial \tau} \psi^\sharp.$$

Lemma 3.2. *Let $U' \Subset U$. If ψ_i , for $i = 1, \dots, v-1$, are smooth functions of a complex variable τ , with $|\psi_i|$ sufficiently small. Then*

$$(3.3) \quad \theta_1 = dz_1 + \psi_1^\sharp(z_n, z_v), \dots, \theta_{v-1} = dz_{v-1} + \psi_{v-1}^\sharp(z_n, z_v), \theta_v = dz_v$$

generate the involutive ideal sheaf \mathcal{J}'_M of a CR structure of type (n, k) on U' .

Proof. The ideal sheaf is generated on U by dz_1, \dots, dz_v . After shrinking, we can assume that $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ are linearly independent on U .

Thus, by taking $|\psi_i|$ sufficiently small, we may keep $\theta_1, \dots, \theta_v, \bar{\theta}_1, \dots, \bar{\theta}_n$ linearly independent in any neighborhood U' of p_0 with $U' \Subset U$. Moreover, since $d\psi_i^\#(z_n, z_v) \wedge dz_v = 0$, for $1 \leq i < v$, we obtain

$$(d\theta_i) \wedge \theta_1 \wedge \dots \wedge \theta_v = d\psi_i^\#(z_n, z_v) \wedge \theta_1 \wedge \dots \wedge \theta_{v-1} \wedge dz_v = 0, \quad \forall i = 1, \dots, v-1.$$

This shows that the ideal sheaf \mathcal{J}'_M generated by $\theta_1, \dots, \theta_v$ is involutive and defines a CR structure of type (n, k) on U' . \square

Let us fix a sequence of distinct complex numbers $\{\tau_j\}$, such that

$$\operatorname{Im} \tau_j > 0 \text{ for all } j, \quad \tau_j \rightarrow 0, \quad \{w_n = \tau_j\} \cap \omega \neq \emptyset \text{ for all } j.$$

For each j we choose an open disk Δ_j in \mathbb{C} , centered at τ_j , and such that $\bar{\Delta}_j \cap \bigcup_{i \neq j} \bar{\Delta}_i = \emptyset$. Provided the τ_j 's are sufficiently close to 0, for each j we can fix a point $w^{(j)} \in \omega$, with $w_n^{(j)} = \tau_j$, and $w^{(j)} \rightarrow 0$, and take the functions ψ_j in Lemma 3.2 in such a way that

$$\begin{aligned} \operatorname{supp} \psi_i &= \bigcup_{j=0}^{\infty} \bar{\Delta}_{i+jv}, \quad \text{for } i = 1, \dots, n, \\ c_{i+jv} &= \int_{A_{i+jv}} \psi_i^\#(z_n, z_v) \wedge dz_n \wedge \dots \wedge dz_{v-1} \wedge dz_v \text{ is real and } > 0, \end{aligned}$$

where (e_1, \dots, e_n) is the canonical basis of \mathbb{C}^n

$$A_j = \pi^{-1}(\{w^{(j)} + (\tau - \tau_j)e_n \mid \tau \in \Delta_j\}).$$

Let u be a CR function on an open neighborhood V of p_0 in U for the structure defined by (3.3). This means that $du_{(p)} \in \mathcal{J}'_{M(p)}$ for all $p \in V$. Since \mathcal{J}_M and \mathcal{J}'_M agree to infinite order outside $\bigcup_j \pi^{-1}(\{w \mid w_n \in \Delta_j\})$, and $\bigcup_j \{w \in \omega \mid w_n \in \Delta_j\}$ does not disconnect ω , by the argument of Lemma 3.1 we have (4.2) for all w in the complement in $\pi(V) \setminus \bigcup_j \{w \in \omega \mid w_n \in \Delta_j\}$. Thus we obtain

$$\begin{aligned} 0 &= \pm \oint_{\tau \in \partial \Delta_j} d\tau \int_{M_{w^{(j)} + \tau e_n}} u dz_n \wedge \dots \wedge dz_{v-1} = \pm \int_{\partial A_j} u dz_n \wedge \dots \wedge dz_v \\ &= \pm \int_{A_j} du \wedge dz_n \wedge \dots \wedge dz_v. \end{aligned}$$

This yields

$$\int_{A_{i+jv}} \frac{\partial u}{\partial z_i} \psi_i^\# \wedge dz_n \wedge \dots \wedge dz_v = 0,$$

where, to compute $\frac{\partial u}{\partial z_i}$, we consider any C^1 -extension of u as a function of the complex variables z_1, \dots, z_v for which $\bar{\partial}u = 0$ at all points of U . Taking the limit, we observe that

$$c_{i+jv}^{-1} \int_{A_{i+jv}} \frac{\partial u}{\partial z_i} \psi_i^\# \wedge dz_n \wedge \dots \wedge dz_v \longrightarrow \frac{\partial u(p_0)}{\partial z_i} \implies \frac{\partial u(p_0)}{\partial z_i} = 0 \quad \forall i = 1, \dots, v-1,$$

which, together with (2.1) shows that $du(p_0) \in \mathbb{C} dz_v(p_0)$.

We have proved:

Theorem 3.3. *Let M be a CR manifold of type (n, k) and assume that M is Lorentzian at a point p_0 . Then we can find a new CR structure of type (n, k) on a neighborhood U of p_0 , which agrees with the original one to infinite order at p_0 , and a real codirection $\eta_0 \in T_{p_0}^* M$ such that, if \mathcal{L} is the distribution of $(0, 1)$ -vector fields for this new structure, all solutions $u \in C^1$ on a neighborhood of p_0 to the homogeneous system (2.1) satisfy $du(p_0) \in \mathbb{C}\eta_0$.*

Proof. Indeed, using Lemma 2.5 we can always reduce to the case in which M is locally embeddable at p_0 . \square

Corollary 3.4. *We can find a new CR structure of type (n, k) on U , which agrees with the original one to infinite order at p_0 , and which is not CR-embeddable at p_0 .*

4. INVOLUTIVE SYSTEMS WHOSE SOLUTIONS ARE CRITICAL AT p_0

In this section we improve the result of the previous section in the case of a Lorentzian CR manifold of the hypersurface type.

We assume that M has CR-dimension $n \geq 2$ and CR-codimension 1, and is Lorentzian and locally embeddable at $p_0 \in M$. We have $m = \dim_{\mathbb{R}} M = n + 2$ and we set $v = n + 1$,

We can fix a CR-chart (U, z_1, \dots, z_v) centered at p_0 , with

$$(4.1) \quad \operatorname{Im} z_v + \sum_{i=2}^v z_i \bar{z}_i = z_1 \bar{z}_1 + O(|z|^3) \quad \text{on } U.$$

By shrinking, we get that $z_1 \bar{z}_1 \geq \frac{1}{2} \operatorname{Im} z_v$ on U . Consider the map $\pi : U \ni p \rightarrow w = (z_2(p), \dots, z_v(p)) \in \mathbb{C}^n$. By a further shrinking, we can assume that there is an open ball $B \subset \mathbb{C}^n$, centered at 0, such that

- $\pi(U) = \bar{\omega}$, with $B \setminus \omega$ strictly convex, and $\partial\omega \cap B$ smooth;
- if $\operatorname{Im} \tau \geq 0$, then $\{w \in B \mid \operatorname{Im} w_n = \tau\} \subset \omega$;
- for all $w \in \omega$ the set $M_w = \pi^{-1}(w)$ is diffeomorphic to the circle S^1 ;
- for $w \in \partial\omega \cap B$ the set $M_w = \pi^{-1}(w)$ is a point.

By repeating the proof of Lemma 3.1, we obtain

Lemma 4.1. *If $u \in \mathcal{O}_M(U)$, then*

$$(4.2) \quad F(w) = \oint_{M_w} u dz_1 = 0, \quad \forall w \in \omega. \quad \square$$

Since $2z_1 \bar{z}_1 \geq \operatorname{Im} z_v$ on U , for any smooth function ψ of a complex variable τ , with $\operatorname{supp} \psi \subset \{\operatorname{Im} \tau \geq 0\}$, the function $z_1^{-1} \psi(z_v)$ can be extended to a smooth function on U , vanishing to infinite order on $\{z_1 = 0\} \cap U$.

Lemma 4.2. *If ψ_i , for $i = 1, \dots, v$ are smooth functions of a complex variable τ , with support contained in $\{\operatorname{Im} \tau \geq 0\}$, then*

$$(4.3) \quad \theta_1 = dz_1 + z_1^{-1} \psi_1(z_v) d\bar{z}_v, \dots, \theta_v = dz_v + z_1^{-1} \psi_v(z_v) d\bar{z}_v$$

(the functions $z_1^{-1} \psi_i(z_v)$ are put = 0 for $z_1 = 0$) generate the ideal sheaf $\mathcal{I}'_{U'}$ of a CR structure of type $(n, 1)$ in a neighborhood U' of p_0 in U , which agrees to infinite order with the original one at p_0 .

Proof. By the condition on the supports, the functions $z_1^{-1} \psi_i(z_v)$ are smooth on U and vanish to infinite for $z_1 = 0$, and in particular at p_0 . Thus $\theta_1, \dots, \theta_v, \bar{\theta}_1, \dots, \bar{\theta}_n$ yield a basis of $\mathbb{C}T_p M$ for p in a suitable neighborhood U' of p_0 , and agree with $dz_1, \dots, dz_v, d\bar{z}_1, \dots, d\bar{z}_n$ to infinite order at p_0 .

We have moreover

$$d\theta_i = z_1^{-1} \frac{\partial \psi_i(z_v)}{\partial z_v} dz_v \wedge d\bar{z}_v - z_1^{-2} \psi_i(z_v) dz_1 \wedge d\bar{z}_v.$$

Hence

$$d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_v = d\theta_i \wedge dz_1 \wedge \cdots \wedge dz_v = 0$$

shows that $\mathcal{F}'_{U'}$ is involutive. The proof is complete. \square

Let us fix a sequence of distinct complex numbers $\{\tau_j\}$, such that

$$\text{Im } \tau_j > 0 \text{ for all } j, \quad \tau_j \rightarrow 0, \quad \{w_n = \tau_j\} \cap \omega \neq \emptyset \text{ for all } j.$$

For each j we choose an open disk Δ_j in \mathbb{C} , centered at τ_j , and such that $\bar{\Delta}_j \cap \bigcup_{i \neq j} \bar{\Delta}_i = \emptyset$. Provided the τ_j 's are sufficiently close to 0, for each j we can fix a point $w^{(j)} \in \omega$, with $w_n^{(j)} = \tau_j$, and $w^{(j)} \rightarrow 0$, and take the functions ψ_j in Lemma 4.2 in such a way that

$$\begin{aligned} \text{supp } \psi_i &= \bigcup_{j=0}^{\infty} \bar{\Delta}_{i+j(v+1)}, \quad \text{for } i = 1, \dots, v, \\ c_{i+j(v+1)} &= \int_{A_{i+j(v+1)}} z_1^{-1} \psi_i(z_v) d\bar{z}_v \wedge dz_1 \wedge dz_v \text{ is real and } > 0, \end{aligned}$$

where

$$A_j = \pi^{-1}(\{w^{(j)} + (\tau - \tau_j)e_n \mid \tau \in \Delta_j\}).$$

Here we denoted by e_1, \dots, e_n the canonical basis of \mathbb{C}^n .

Let u be a CR function on an open neighborhood V of p_0 in U' for the structure defined by (5.2). This means that $du_{(p)} \in \mathcal{F}'_{U',(p)}$ for all $p \in V$. Since \mathcal{F}_M and $\mathcal{F}'_{U'}$ agree to infinite order on $\pi(U')$ outside $\bigcup_i \text{supp } \psi_i(w)$, and this set does not disconnect U , by the argument of Lemma 3.1 we have (4.2) for all w in the complement in $\pi(V)$ of $\bigcup_j \{w \in \omega \mid w_n \in \Delta_j\}$. Thus we obtain

$$0 = \pm \oint_{\tau \in \partial \Delta_j} d\tau \oint_{M_{w^{(j)} + \tau e_n}} u dz_n = \pm \int_{\partial A_j} u dz_1 \wedge dz_v = \pm \int_{A_j} du \wedge dz_1 \wedge dz_v.$$

This yields

$$(4.4) \quad I_{i+j(v+1)}(u) = \int_{A_{i+j(v+1)}} \frac{\partial u}{\partial z_i} z_1^{-1} \wedge d\bar{z}_v \wedge dz_1 \wedge dz_v = 0,$$

where, to compute $\frac{\partial u}{\partial z_i}$, we consider any C^1 -extension of u as a function of the complex variables z_1, \dots, z_v for which $\bar{\partial}u = 0$ at all points of V . When $j \rightarrow \infty$, $c_{i+j(v+1)}^{-1} I_{i+j(v+1)} \rightarrow \frac{\partial u(p_0)}{\partial z_i}$. Hence, from (4.4) we obtain that $\frac{\partial u(p_0)}{\partial z_i} = 0$ for $1 \leq i \leq v$, which, together with (2.1) shows that $du(p_0) = 0$.

We have proved:

Theorem 4.3. *If M is a CR manifold of type $(n, 1)$ and is Lorentzian at $p_0 \in M$, then we can find a new CR structure of type $(n, 1)$ on an open neighborhood U of p_0 in M , which agrees with the original one to infinite order at p_0 , such that, if \mathcal{Z} is the distribution of $(0, 1)$ -vector fields for this new structure, all solutions $u \in C^1$ on a neighborhood of p_0 to the homogeneous system (2.1) satisfy $du(p_0) = 0$.*

Proof. We can apply the discussion above after reducing, by Lemma 2.5, to the case in which M is locally CR-embeddable at p_0 . \square

5. THE CASE OF HIGHER CODIMENSION

In this section we extend the result of Theorem 4.3 to some CR manifolds with CR dimension and CR codimension both greater than 1. To this aim we will first recall some results on weak unique continuation and next consider morphisms of CR manifolds.

5.1. Minimal locally CR -embeddable CR manifolds and unique continuation.

We recall that a CR submanifold M is *minimal* at $p_0 \in M$ if there is no germ (N, p_0) of CR submanifold of M at p_0 , having the same CR dimension, but smaller CR codimension. We have

Lemma 5.1. *Assume that M is minimal and locally CR -embeddable at $p_0 \in M$.*

Let (S, p_0) be a germ of a CR submanifold of M , of type $(0, \nu)$. Then a germ $f \in \mathcal{O}_{M, (p_0)}$ of a CR function at p_0 , vanishing on (S, p_0) , is equal to 0.

If M is minimal and locally CR -embeddable at all points, then the CR functions on M satisfy the weak unique continuation principle.

Proof. In the first part of the proof, we can assume that M is a generic CR submanifold of an open set in \mathbb{C}^ν . For any open neighborhood U of p_0 in M , there are an open neighborhood U_0 of p_0 in U , and an open wedge W in \mathbb{C}^ν , with edge U_0 , such that, the restriction $u|_{U_0}$ of any $u \in \mathcal{O}_M(U)$ is the boundary value of a holomorphic function \tilde{u} , defined on W (see [42, 43, 6]). Assume now that $u \in \mathcal{O}_M(U)$ vanishes on S . Then $\tilde{u} = 0$ by the edge of the wedge theorem (see [38]), and therefore $u = 0$.

The last statement follows by unique continuation for holomorphic functions on open subsets of \mathbb{C}^ν . \square

5.2. CR -maps with simple singularities. Let M, N be CR manifolds. A smooth map $\pi : M \rightarrow N$ is CR if $d\pi(T^{0,1}M) \subset T^{0,1}N$. We say that π is

- a CR -immersion if $\ker d\pi = 0$ and $d\pi(T^{0,1}M) = d\pi(\mathbb{C}TM) \cap T^{0,1}N$;
- a CR -submersion if $d\pi(T_p M) = T_{\pi(p)}N$ and $d\pi(T_p^{0,1}M) = T_{\pi(p)}^{0,1}N$, $\forall p \in M$;
- a local CR -diffeomorphism if it is at the same time a CR -immersion and a CR -submersion.

Next we consider critical points of some CR -maps.

Let $k \geq 1$ and $\pi : M \rightarrow N$ a CR -map, with M of type (n, k) and N of type $(n, k-1)$.

If $p_0 \in M$ is not a critical point of π , then π is a CR -submersion near p_0 .

Assume now that p_0 is a critical point of π , and $q_0 = \pi(p_0)$ the corresponding critical value. Then the rank of $d\pi(p_0)$ is less than $2n + k - 1$. Assume that it is exactly equal to $2n + k - 2$. Then the dual map $d\pi^*(p_0) : T_{q_0}^*N \rightarrow T_{p_0}^*M$ is not injective, and has a 1-dimensional kernel.

Definition 5.2. If $\ker d\pi^*(p_0) \cap H_{q_0}^0N = \{0\}$, we say that π has at p_0 a CR -noncharacteristic singularity.

Assume that this is the case and fix $0 \neq \eta_0 \in \ker d\pi^*(p_0)$. Then there is η'_0 , uniquely determined modulo $H_{q_0}^0N$, such that $\eta_0 + i\eta'_0 \in \mathbf{Z}_{q_0}^0N$, and we obtain an element $\xi_0 \in H_{p_0}^0M$, with $0 \neq \xi_0 = d\pi^*(p_0)(\eta'_0)$.

Definition 5.3. If we can choose η'_0 in such a way that \mathbf{L}_{ξ_0} has 1 positive and $n-1$ negative eigenvalues, we say that π has a *Lorentzian CR -non characteristic singularity* at p_0 .

Assume now that M and N are locally CR -embeddable at p_0, q_0 , respectively, and that L_{ξ_0} has 1 positive and $(n-1)$ negative eigenvalues. We set $v = n + k$. We can choose CR -charts $(U; z_1, \dots, z_v)$ of M , centered at p_0 , and $(W; w_2, \dots, w_v)$ of N , centered at q_0 , with $\pi(U) \subset W$ and $z_j = \pi^* w_j$ for $j = 2, \dots, v$, such that $i\xi_0 = dz_v(p_0)$, and

$$\operatorname{Im} z_v = h(z) \quad \text{on } U, \text{ with } h(z) = z_1 \bar{z}_1 - \sum_{i=2}^v z_i \bar{z}_i + O(|z|^3).$$

Lemma 5.4. *Let $D = \{p \in U \mid z_1(p) = 0\}$. Then, there is an open neighborhood U' of p_0 in U , an open neighborhood ω of q_0 in N , and an open domain ω_- in ω , with $q_0 \in \partial\omega_-$, such that*

- (1) $\omega_- \subset \pi(U') \subset \omega$, $\pi(D \cap U') \subset \partial\omega$ and ω is strictly pseudoconcave at q_0 ;
- (2) $\pi : U' \rightarrow N$ is proper and, for $q \in \pi(U')$, $\pi^{-1}(q)$ is either a point or is diffeomorphic to a circle.

Proof. Provided U is sufficiently small, the restriction of π to D is a smooth diffeomorphism of D onto a closed hypersurface $\pi(D)$ in an open neighborhood ω of q_0 in N . By further shrinking, we can assume that $A \setminus \pi(D)$ consists of two connected components ω_+ and ω_- and that $\omega_- \subset \pi(U)$.

Since $\omega_- = \{\operatorname{Im} w_v + \sum_{i=2}^v w_i \bar{w}_i + O(|w|^3) > 0\}$ near q_0 , we have $q_0 \in \partial\omega_-$ and ω_- strictly pseudoconcave at q_0 . Moreover, by taking U small, we can assume that

$$\operatorname{Im} z_v + \frac{3}{2} \sum_{i=2}^v z_i \bar{z}_i \geq \frac{1}{2} z_1 \bar{z}_1 \quad \text{on } U,$$

and therefore we obtain an U' satisfying (1) and (2) by setting $U' = U \cap \pi^{-1}(\omega)$ for a smaller neighborhood ω of q_0 in N . \square

5.3. Perturbation of the CR structure of M . We keep the notation of §5.2, and we shall assume that (1) and (2) of Lemma 5.4 hold true with $U' = U$.

Lemma 5.5. *Assume that N is a minimal CR manifold. If u is a CR function on a connected open neighborhood V of p_0 in U , then*

$$(5.1) \quad g(q) := \oint_{\pi^{-1}(q)} u dz_1 = 0, \quad \forall q \in \pi(V).$$

Proof. First we note that $W = \pi(V) \cup (\omega \setminus \pi(U))$ is a neighborhood of q_0 in N . The function g , equal to the left hand side of (5.1) for $w \in \pi(V)$ and 0 on $W \setminus \pi(V)$ is continuous, because the fiber $\pi^{-1}(q)$ shrinks to a point when $q \rightarrow \partial\pi(V) \cap \omega$. Since $\pi(V)$ is connected and its connected component in W contains an open subset where $g = 0$, our contents follows by the weak unique continuation principle (see Lemma 5.1) if we show that g is a CR function on W . To this aim, it suffices to show that

$$\int_N dg \wedge \eta = 0, \quad \forall \eta \in \Omega_0^{2n+k-2}(W) \cap \mathcal{J}_N^{n+k-1}(W),$$

where $\Omega_0^*(W)$ means smooth exterior forms with compact support in W . We note that $\pi^* \eta \in \Omega_0^{2n+k-2}(V) \cap \mathcal{J}_M^{n+k-1}(W)$, because the map π is CR and proper. Thus we obtain

$$\int_N dg \wedge \eta = \int_M du \wedge dz_1 \wedge \pi^* \eta = 0,$$

because u is CR on a neighborhood of the support of $dz_1 \wedge \pi^* \eta \in \Omega_0^{2n+k-2}(V) \cap \mathcal{J}_M^{n+k}(V)$. The proof is complete. \square

By shrinking, we get $2z_1\bar{z}_1 \geq \text{Im } z_v$ on U . In particular, if ψ is a smooth function of one complex variable τ , with $\text{supp } \psi \subset \{\text{Im } \tau \geq 0\}$, the function $z_1^{-1}\psi(z_v)$ can be extended to a smooth function on U , vanishing to infinite order on $\{z_1 = 0\} \cap U$.

Lemma 5.6. *If ψ_i , for $i = 1, \dots, v$ are smooth functions of a complex variable τ , with support contained in $\{\text{Im } \tau \geq 0\}$, then*

$$(5.2) \quad \theta_1 = dz_1 + z_1^{-1}\psi_1(z_v)d\bar{z}_v, \dots, \theta_v = dz_v + z_1^{-1}\psi_v(z_v)d\bar{z}_v$$

(the functions $z_1^{-1}\psi_i(z_v)$ are put = 0 for $z_1 = 0$) generate the ideal sheaf $\mathcal{J}'_{U'}$ of a CR structure of type (n, k) in a neighborhood U' of p_0 in U , which agree to infinite order with the original one at p_0 .

Proof. By the condition on the supports, the functions $z_1^{-1}\psi_i(z_v)$ are smooth on U and vanishing to infinite for $z_1 = 0$, and in particular at p_0 . Thus $\theta_1, \dots, \theta_v, \bar{\theta}_1, \dots, \bar{\theta}_n$ yield a basis of $\mathbb{C}T_p M$ for p in a suitable neighborhood U' of p_0 , and agree with $dz_1, \dots, dz_v, d\bar{z}_1, \dots, d\bar{z}_n$ to infinite order at p_0 . We have moreover

$$d\theta_i = z_1^{-1} \frac{\partial \psi_i(z_v)}{\partial z_v} dz_v \wedge d\bar{z}_v - z_1^{-2} \psi_i(z_v) dz_1 \wedge d\bar{z}_v.$$

Hence

$$d\theta_i \wedge \theta_1 \wedge \dots \wedge \theta_v = d\theta_i \wedge dz_1 \wedge \dots \wedge dz_v = 0$$

shows that $\mathcal{J}'_{U'}$ is involutive. The proof is complete. \square

Let us fix a sequence of distinct complex numbers $\{\tau_j\}$, such that

$$\text{Im } \tau_j > 0 \text{ for all } j, \quad \tau_j \rightarrow 0, \quad \{w_n = \tau_j\} \cap \omega \neq \emptyset \text{ for all } j.$$

For each j we choose an open disk Δ_j in \mathbb{C} , centered at τ_j , in such a way that $\bar{\Delta}_j \cap \bigcup_{i \neq j} \bar{\Delta}_i = \emptyset$. Next we choose balls B_j in \mathbb{C}^{v-2} with

$$K_j = \{q \in \omega \mid (w_2(q), \dots, w_{v-1}(q)) \in B_j, \quad w_v \in \bar{\Delta}_j\} \subseteq \omega.$$

We set $\partial_0 K_j = \{q \in K_j \mid w_v(q) \in \partial \Delta_j\}$, $A_j = \pi^{-1}(K_j)$. Note that the A_j 's are compact because π is proper.

Then we take the functions ψ_i in Lemma 5.6 in such a way that, for suitable forms $\eta_j \in \mathcal{Q}_0^{n-1}(B_j)$, we have

$$\begin{aligned} \text{supp } \psi_i &= \bigcup_{j=0}^{\infty} \bar{\Delta}_{i+j(v+1)}, \quad \text{for } i = 1, \dots, v, \\ c_{i+j(v+1)} &= \int_{A_{i+j(v+1)}} z_1^{-1} \psi_i(z_v) d\bar{z}_v \wedge dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \eta_j \text{ is real and } > 0. \end{aligned}$$

Let u be a CR function on a connected open neighborhood V of p_0 in U' for the structure defined by (5.2). Since \mathcal{J}_M and $\mathcal{J}'_{U'}$ agree to infinite order on $\pi(U')$ outside $E = \{w_v \in \bigcup_i \text{supp } \psi_i\}$, and this set does not disconnect $\pi(V)$, by the argument of Lemma 5.5 we have (5.1) for all q in the complement in $\pi(V)$ of E . Thus we obtain

$$\begin{aligned} 0 &= \int_{\partial K_j} \left(\oint_{\pi^{-1}(q)} u dz_1 \right) dw_2 \wedge \dots \wedge dw_v \wedge \eta_j = \int_{\partial A_j} u dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \eta_j \\ &= \int_{A_j} du \wedge dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \eta_j, \quad \text{yielding} \\ (5.3) \quad &\int_{A_{i+j(v+1)}} \frac{\partial u}{\partial z_i} z_1^{-1} \psi_i(z_v) d\bar{z}_v \wedge dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \eta_{i+j(v+1)} = 0, \end{aligned}$$

where, to compute $\frac{\partial u}{\partial z_i}$, we consider any C^1 -extension of u as a function of the complex variables z_1, \dots, z_v for which $\bar{\partial}u = 0$ at all points of V . But

$$c_{i+j(v+1)}^{-1} \int_{A_i+j(v+1)} \frac{\partial u}{\partial z_i} z_1^{-1} \psi_i(z_v) d\bar{z}_v \wedge dz_1 \wedge \dots \wedge dz_v \wedge \pi^* \eta_{i+j(v+1)}$$

converges to $\frac{\partial u(p_0)}{\partial z_i}$ when $j \rightarrow +\infty$. Therefore $\frac{\partial u(p_0)}{\partial z_i} = 0$ for $i = 1, \dots, v$. Together with (2.1), this shows that $du(p_0) = 0$.

We have proved:

Theorem 5.7. *Let M, N , be CR manifolds of types (n, k) and $(n, k-1)$, respectively, with $k \geq 1$. Assume that N is minimal and that there is a CR map $\pi : M \rightarrow N$ having a Lorentzian CR-non characteristic singularity at $p_0 \in M$. Then we can find a new CR structure of type (n, k) on an open neighborhood U of p_0 in M , which agrees with the original one to infinite order at p_0 , such that, if \mathcal{L} is the distribution of $(0, 1)$ -vector fields for this new structure, all solutions $u \in C^1$ on a neighborhood of p_0 to the homogeneous system (2.1) satisfy $du(p_0) = 0$.*

Proof. The discussion above proves the theorem in the case where both M and N are locally CR-embeddable at p_0 and at $q_0 = \pi(p_0)$, respectively. In general, we can reduce to this case by taking formal power series solutions $\{z\}_1, \dots, \{z\}_v$ at p_0 , $\{w\}_2, \dots, \{w\}_v$ at q_0 , to the homogeneous tangential Cauchy-Riemann systems on M and N , respectively, with $\{z_j\} = \pi^*\{w\}_j$ for $j = 2, \dots, v$. Then we take smooth functions w_2, \dots, w_v on N having Taylor series $\{w\}_2, \dots, \{w\}_v$ at q_0 , define $z_j = \pi^*w_j$ for $j = 2, \dots, v$, and choose a smooth function z_1 on M with Taylor series $\{z_1\}$ at p_0 . By restricting to a suitable neighborhood U of p_0 in M and W of q_0 in N , we obtain CR-charts $(U; z_1, \dots, z_v)$ and $(W; w_2, \dots, w_v)$ for new CR structures which agree to infinite order with the original ones at p_0 in M and at q_0 in N . The same map π has a Lorentzian CR-noncharacteristic singularity at p_0 also for the new locally CR-embeddable CR structures, so that the previous discussion applies. \square

5.4. Example. Let $n \geq 1$, $k \geq 1$, $v = n + k$, and N any CR manifold of type $(n, k-1)$, contained in an open neighborhood G of 0 in \mathbb{C}^{v-1} , and minimal. Let w_1, \dots, w_{v-1} be the canonical holomorphic coordinates of \mathbb{C}^{v-1} and assume that dw_1 and $d\bar{w}_1$ are linearly independent on N .

Let $\phi : \mathbb{C}^v \ni (z_1, \dots, z_v) \rightarrow (z_1, \dots, z_{v-1}) \in \mathbb{C}^{v-1}$ be the projection onto the first $v-1$ coordinates. If

$$M = \{z \in \mathbb{C}^v \mid \phi(z) \in N, \operatorname{Im} z_1 + \sum_{i=1}^{v-1} z_i \bar{z}_i = z_v \bar{z}_v\},$$

then M is a minimal CR submanifold of type (n, k) of $\pi^{-1}(G) \subset \mathbb{C}^v$, and the restriction of ϕ describes a CR map $\pi : M \rightarrow N$ which has at 0 a Lorentzian CR-noncharacteristic singularity.

In particular, there are CR structures on a minimal Lorentzian CR manifold M of arbitrary CR codimension such that a point $p_0 \in M$ is critical for all CR functions defined on a neighborhood of p_0 .

6. TANGENTIAL CAUCHY-RIEMANN COMPLEXES AND A GLOBAL EXAMPLE

Throughout this section, M is a smooth CR manifold, of positive CR dimension n , and arbitrary CR codimension $k \geq 0$. We set $v = n + k$, and denote by \mathcal{Z} the distribution of smooth complex vector fields of type $(0, 1)$ on M .

We recall that $\mathcal{O}_M(U)$ is the space of CR functions of class C^1 on $U^{\text{open}} \subset M$. We set $\mathcal{O}_M^\infty(U) = \mathcal{O}_M(U) \cap C^\infty(U)$, and $\mathcal{O}_M^\infty(\bar{U}) = \mathcal{O}_M(U) \cap C^\infty(\bar{U})$ for the restrictions to \bar{U} of smooth functions on M , which are CR in U . Likewise, when Ω is an open subset of a complex manifold \mathbb{X} , we write $\mathcal{O}(\bar{\Omega})$ for $C^\infty(\bar{\Omega}) \cap \mathcal{O}(\Omega)$.

6.1. Definition of the $\bar{\partial}_M$ -complexes. Let \mathcal{J}_M be the ideal sheaf, corresponding to the characteristic distribution \mathcal{Z} of $(0, 1)$ -vector fields on M . Formal integrability of \mathcal{Z} is equivalent (see Lemma 3.2) to

$$(6.1) \quad d\mathcal{J}_M \subset \mathcal{J}_M.$$

This implies that also $d(\mathcal{J}_M)^a \subset (\mathcal{J}_M)^a$, where, for each positive integer a , $(\mathcal{J}_M)^a$ is the a -th exterior power of the ideal \mathcal{J}_M , and we set $(\mathcal{J}_M)^0 = \mathcal{O}_M^*$. We can define cochain complexes on M by considering the quotients $\mathcal{Q}_M^{a,*} = (\mathcal{J}_M)^a / (\mathcal{J}_M)^{a+1}$ and the map $\bar{\partial}_M : \mathcal{Q}_M^{a,*} \rightarrow \mathcal{Q}_M^{a+1,*}$ induced on the quotients by the exterior differential. We have

$$\mathcal{Q}_M^{a,*} = \bigoplus_{q \geq 0} \mathcal{Q}_M^{a,q}, \quad \text{with } \mathcal{Q}_M^{a,q} = ((\mathcal{J}_M)^a \cap \Omega_M^{a+q}) / ((\mathcal{J}_M)^{a+1} \cap \Omega_M^{a+q}),$$

and $\bar{\partial}_M(\mathcal{Q}_M^{a,q}) \subset \mathcal{Q}_M^{a,q+1}$. This indeed was the intrinsic definition for the tangential Cauchy-Riemann complexes on CR manifolds given in [32].

Let $\mathcal{Z}^{a,q}(U) = \{f \in \mathcal{Q}_M^{a,q}(U) \mid \bar{\partial}_M f = 0\}$ and $\mathcal{B}^{a,q}(U) = \bar{\partial}_M(\mathcal{Q}_M^{a,q-1}(U))$. The quotient $H^{a,q}(U) = \mathcal{Z}^{a,q}(U) / \mathcal{B}^{a,q}(U)$ is the *cohomology group* of the smooth cohomology of $\bar{\partial}_M$ on U in bidegree (a, q) . We set

$$H^{a,q}(p_0) = \lim_{\longrightarrow p_0 \in U^{\text{open}}} H^{a,q}(U)$$

for the group of germs of bidegree (a, q) -cohomology classes at p_0 .

Let us give a more explicit description of the equations involved in the $\bar{\partial}_M$ -complexes.

An element of $\mathcal{Z}^{a,q}(U)$ has a representative $f \in \mathcal{Q}_M^{p+q}(U)$. The conditions that $f \in (\mathcal{J}_M)^a$ and that its class $[f] \in \mathcal{Z}^{a,q}(U)$ satisfies the integrability condition $\bar{\partial}_M[f] = 0$ are expressed by

$$(6.2) \quad \begin{cases} f \wedge \eta_1 \wedge \cdots \wedge \eta_{v-a+1} = 0, \quad \forall \eta_1, \dots, \eta_{v-a+1} \in \mathcal{J}_M^1(U), \\ df \wedge \eta_1 \wedge \cdots \wedge \eta_{v-a+1} = 0, \quad \forall \eta_1, \dots, \eta_{v-a+1} \in \mathcal{J}_M^1(U), \end{cases}$$

and the equation $\bar{\partial}_M \alpha = [f]$ for $\alpha \in \mathcal{Q}_M^{a,q-1}(U)$ is equivalent to finding $u \in \mathcal{Q}_M^{a+q-1}(U)$ such that

$$(6.3) \quad \begin{cases} u \wedge \eta_1 \wedge \cdots \wedge \eta_{v-a+1} = 0, \quad \forall \eta_1, \dots, \eta_{v-a+1} \in \mathcal{J}_M^1(U), \\ (du - f) \wedge \eta_1 \wedge \cdots \wedge \eta_{v-a+1} = 0, \quad \forall \eta_1, \dots, \eta_{v-a+1} \in \mathcal{J}_M^1(U). \end{cases}$$

Both equation (6.2) and (6.3) are meaningful when f, u are currents, and therefore we can consider the $\bar{\partial}_M$ -complexes on currents, or require different degrees of regularity on the data and the solution.

6.2. Absence of Poincaré lemma. In general, the $\bar{\partial}_M$ -complexes are not acyclic (see e.g. [3, 4, 18, 32]). In fact, the perturbations we used in the previous section to deduce non complete-integrability results utilize elements of $H^{0,1}(p_0)$. The arguments of §4 provide simpler proofs of the absence of the Poincaré lemma in some special cases. We have e.g. (see §4)

Proposition 6.1. *Let M be a CR manifold of type $(n, 1)$, which is locally CR-embeddable and Lorentzian at p_0 , and let $(U; z_1, \dots, z_v)$, with $v = n + 1$, be a CR chart centered at p_0 for which (4.1) holds and $2z_1\bar{z}_1 \geq \text{Im } z_v$ on U .*

Let ψ be a smooth function of one complex variable τ , with compact support contained in $\{\text{Im } \tau \geq 0\}$. Then $\omega = z_1^{-1}\psi(z_v)d\bar{z}_v$, continued by 0 where $z_1 = 0$, defines a smooth $\bar{\partial}_M$ -closed 1-form. A necessary and sufficient condition for ω to be cohomologous to 0 is that

$$(6.4) \quad \iint \tau^h \psi(\tau) d\tau \wedge d\bar{\tau} = 0, \quad \forall h \in \mathbb{Z}, \quad h \geq 0.$$

Proof. Assume indeed that there is $u \in C^1(U)$ such that $\bar{\partial}_M u = [\omega]$. We keep the notation of §4 for π , ω , B , and use integration on the fiber to define

$$g(w) = \frac{-1}{2\pi i} \oint_{M_w} u dz_1, \quad \text{i.e.} \quad \int_{\omega} g\phi = \int_U u \wedge dz_1 \wedge \pi^* \phi, \quad \forall \phi \in \Omega^{2n}.$$

Clearly $g = 0$ on $\partial\omega \cap B$ and furthermore we obtain

$$\begin{aligned} \int_{\omega} dg \wedge \eta &= \frac{-1}{2\pi i} \int_U du \wedge dz_1 \wedge \pi^* \eta \\ &= \frac{-1}{2\pi i} \int z_1^{-1} \psi(z_v) d\bar{z}_v \wedge dz_1 \wedge \pi^* \eta = \int_{\omega} \psi(w_n) d\bar{w}_n \wedge \eta \\ &\quad \forall \eta \in \Omega_0^{n-1, n}(\omega). \end{aligned}$$

Thus g satisfies

$$\begin{cases} \bar{\partial}g = \psi(w_n) d\bar{w}_n, & \text{on } \partial\omega, \\ g = 0, & \text{on } \partial\omega \cap B. \end{cases}$$

These equations imply that $\tau \rightarrow g(0, \tau)$ has compact support and hence that the equation $\partial v / \partial \bar{\tau} = \psi(\tau)$ has a solution with compact support. This is equivalent to the momentum conditions (6.4) in the statement. \square

We can repeat the same argument to show that the $\bar{\partial}_M$ complexes are not acyclic in dimension q when M is strictly q -pseudoconvex at p_0 . Still restraining to type $(n, 1)$ this condition means that, for a suitable CR chart $(U; z_1, \dots, z_v)$ centered at p_0 we have

$$(6.5) \quad \text{Im } z_v + \sum_{i=q+1}^v z_i \bar{z}_i = \sum_{i=1}^q z_i \bar{z}_i + O(|z|^3) \quad \text{on } U.$$

Here $v = n + 1$. By taking U small we can assume that $\text{Im } z_v \leq 2 \sum_{i=1}^q z_i \bar{z}_i$ on U . Set $\ell = v - q$. By taking U sufficiently small we obtain a proper map

$$(6.6) \quad \pi : U \ni p \rightarrow (z_{q+1}(p), \dots, z_v(p)) \in \omega \subset \mathbb{C}^{\ell},$$

with ω the complement of a strictly convex open subset in an open ball B of \mathbb{C}^{ℓ} , and $M_w \sim S^{2q-1}$ for $w \in \dot{\omega}$ and $M_w \sim$ a point for $w \in \partial\omega \cap B$. Let

$$K_{q-1}(z_1, \dots, z_q) = \frac{\sum_{i=1}^q \bar{z}_i \omega_i(z_1, \dots, z_q)}{(\sum_{i=1}^q z_i \bar{z}_i)^q} \quad \text{with} \quad \omega_i(z_1, \dots, z_q) = \frac{\partial}{\partial \bar{z}_i} \rfloor d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q.$$

Proposition 6.2. *Let M be a CR manifold of type $(n, 1)$, which is locally CR-embeddable and strictly q -pseudoconvex at p_0 . Take a CR-chart (U, z_1, \dots, z_n) with (6.5), $\text{Im } z_n \leq 2z_1 \bar{z}_1$ on U , and let $\pi : M \rightarrow \omega$ be given by (6.6).*

If ψ is a smooth function with compact support of one complex variable τ , with $\text{supp } \psi \subset \{\text{Im } \tau \geq 0\}$, then $\omega = \psi(z_n) d\bar{z}_n \wedge K_{q-1}(z_1, \dots, z_q)$, continued by 0 where $z_1 = 0$, defines a smooth $\bar{\partial}_M$ -closed q -form on U , and (6.4) is a necessary and sufficient condition for ω to be cohomologous to 0.

Proof. The proof is analogous to that of Proposition 6.1. Here we need to utilize, instead of Cauchy's formula, the identity

$$(6.7) \quad \int_{\pi^{-1}(w)} K_{q-1}(z_1, \dots, z_q) \wedge dz_1 \wedge \dots \wedge dz_q = \frac{(-1)^{q(q-1)/2} (q-1)!}{(2\pi i)^q},$$

$$\forall w \in \dot{\omega} \cap \text{supp } \psi(w_\ell).$$

We have indeed $d(K_{q-1}(z_1, \dots, z_q) \wedge dz_1 \wedge \dots \wedge dz_q) = 0$ and then the value of the left hand side of (6.7) is a homology invariant. For all $w \in \dot{\omega}$, the fiber $\pi^{-1}(w)$ is equivalent to the sphere S^{2q-1} . Then the value in (6.7) can be computed by integrating on S^{2q-1} (see [24]). \square

6.3. Strictly pseudoconvex subdomains of CR manifolds. Let Ω be an open set in M . Saying that $\partial\Omega$ is smooth at a point $p_0 \in \partial\Omega$ means that there is an open neighborhood U of p_0 in M and a smooth real valued function $\phi \in C^\infty(U, \mathbb{R})$ such that

$$(6.8) \quad \Omega \cap U = \{p \in U \mid \phi(p) < 0\}, \quad d\phi(p_0) \neq 0.$$

Definition 6.3. We say that Ω is *strictly pseudoconvex* at p_0 if $\mathfrak{L}_{d\phi(p_0)} > 0$ on $\mathbf{Z}_{p_0}M \cap \ker d\phi(p_0)$.

Note that $\mathbf{Z}_{p_0}M \cap \ker d\phi(p_0) = \mathbf{Z}_{p_0}M$ when $d\phi(p_0) \in H_{p_0}^0 M$.

We need to introduce a weaker notion of CR function on M .

Definition 6.4. If $U^{\text{open}} \subset M$, we say that $u \in C^0(U)$ is CR if

$$\int_U u d\eta = 0, \quad \forall \eta \in (\mathcal{J}_M)^\vee(U) \cap \mathcal{Q}_{M,0}^{2n+k-1}(U),$$

where we indicate by $\mathcal{Q}_{M,0}^*(U)$ smooth exterior forms having compact support in U .

If M is a CR submanifold of a complex manifold \mathbb{X} and is minimal at p_0 , then all germs of continuous CR functions extend holomorphically to wedges in \mathbb{X} whose edges contain neighborhoods of p_0 (see [42, 43]). We set $\mathcal{O}_M^{\text{cont}}$ for the sheaf of germs of continuous CR functions on M . For every $U^{\text{open}} \subset M$, the space $\mathcal{O}_M^{\text{cont}}(U)$ is Fréchet for the topology of uniform convergence on the compact subsets of U .

Definition 6.5. Let Ω be an open subset of M , $f \in \mathcal{O}_M(\Omega)$, and $p_0 \in \partial\Omega$. We say that f weakly CR-extends beyond p_0 if there exists a connected open neighborhood U of p_0 in M and $g \in \mathcal{O}_M^{\text{cont}}(U)$ such that $\{p \in \Omega \cap U \mid g(p) = f(p)\}$ has a non empty interior.

Proposition 6.6. *Let M be a CR submanifold of CR-dimension $n \geq 1$ and arbitrary CR codimension $k \geq 0$ of a Stein manifold \mathbb{X} , and Ω a smooth strictly pseudoconvex domain in \mathbb{X} , with $\partial\Omega \cap M \neq \emptyset$. Let M_0 be the set of points of $M \cap \partial\Omega$ at which M is minimal. Consider on $\mathcal{O}(\bar{\Omega})$ the natural Fréchet topology of uniform convergence with all derivatives on the compact subsets of $\bar{\Omega}$.*

Then the set \mathbb{E} of the elements $f \in \mathcal{O}(\bar{\Omega})$ such that $f|_{M \cap \bar{\Omega}}$ weakly CR-extends beyond some point $p \in M_0$ is of the first Baire category in $\mathcal{O}(\bar{\Omega})$.

Proof. Fix a point $p_0 \in M_0$ and a countable fundamental system of open neighborhoods $\{V_\nu\}_{\nu \in \mathbb{N}}$ of p_0 in \mathbb{X} . We can assume that for every $\nu \in \mathbb{N}$ the intersection $M \cap V_\nu$ is connected and contained in the set of minimal points of M . In particular, two CR functions on $M \cap V_\nu$ which agree on some non empty open subset of $M \cap V_\nu$, are equal on all $M \cap V_\nu$. For each ν the set

$$\mathbb{F}_\nu = \{(f, g) \in \mathcal{O}(\bar{\Omega}) \times \mathcal{O}_M^{\text{cont}}(M \cap V_\nu) \mid g = f \text{ on } M \cap \Omega \cap V_\nu\}$$

is a Fréchet space, being a closed subspace of $\mathcal{O}(\bar{\Omega}) \times \mathcal{O}_M^{\text{cont}}(M \cap V_\nu)$.

Let $\pi_\nu : \mathbb{F}_\nu \rightarrow \mathcal{O}(\bar{\Omega})$ be the projection into the first component.

Assume by contradiction that the set of $f \in \mathcal{O}(\bar{\Omega})$ such that $f|_{M \cap \Omega}$ can be weakly CR-continued beyond p_0 is of the second Baire category. Then $\bigcup_{\nu \in \mathbb{N}} \pi_\nu(\mathbb{F}_\nu)$ is of the second Baire category, and hence at least one $\pi_{\nu_0}(\mathbb{F}_{\nu_0})$ is of the second Baire category in $\mathcal{O}(\bar{\Omega})$. Hence $\pi_{\nu_0}(\mathbb{F}_{\nu_0}) = \mathcal{O}(\bar{\Omega})$ and, by Banach-Schauder's theorem, the map $\pi_{\nu_0} : \mathbb{F}_{\nu_0} \rightarrow \mathcal{O}(\bar{\Omega})$ is open (see e.g [39, §2.1]). By the assumption of minimality and the fact that Lemma 5.1 also applies to continuous CR functions, for each $f \in \mathcal{O}(\bar{\Omega})$ there is at most one $g \in \mathcal{O}_M^{\text{cont}}(M \cap V_{\nu_0})$ such that $(f, g) \in \mathbb{F}_{\nu_0}$.

Thus, we conclude that there is a relatively compact open neighborhood V of p_0 in \mathbb{X} such that

$$(6.9) \quad \forall f \in \mathcal{O}(\bar{\Omega}) \quad \exists! g \in \mathcal{O}_M(M \cap V) \quad \text{with } f = g \text{ on } M \cap \Omega \cap V \text{ and}$$

$$(6.10) \quad |g(p)| \leq \|f\|_{\ell, K}, \quad \forall p \in M \cap V,$$

where K is a compact subset of $\bar{\Omega}$ and $\|f\|_{\ell, K}$ a seminorm involving the derivatives of f up to order ℓ on K . Let K_ν be a sequence of compact subsets of \mathbb{X} such that $K_{\nu+1} \subseteq \text{int}(K_\nu)$, for all $\nu \in \mathbb{N}$, and $\bigcap_\nu K_\nu = K$. By Cauchy's inequalities, there are constants $C_\nu > 0$ such that

$$\|f\|_{\ell, K} \leq C_\nu \sup_{K_\nu} |f|, \quad \forall f \in \mathcal{O}(\mathbb{X}).$$

Using (6.10) we obtain

$$|f(p)| \leq C_\nu \sup_{K_\nu} |f|, \quad \forall f \in \mathcal{O}(\mathbb{X}), \quad \forall p \in M \cap V.$$

By applying this inequality to the positive integral powers of the entire functions on \mathbb{X} , we obtain that

$$\begin{aligned} |f(p)| &\leq C_\nu^{1/h} \sup_{K_\nu} |f|, \quad \forall f \in \mathcal{O}(\mathbb{X}), \quad \forall p \in M \cap V, \quad \forall 0 < h \in \mathbb{N} \\ \implies |f(p)| &\leq \sup_{K_\nu} |f|, \quad \forall f \in \mathcal{O}(\mathbb{X}), \quad \forall p \in M \cap V. \end{aligned}$$

Since $\bigcap_\nu K_\nu = K$, we obtain

$$|f(p)| \leq \sup_K |f|, \quad \forall f \in \mathcal{O}(\mathbb{X}).$$

But this gives a contradiction, because the fact that Ω is strictly pseudoconvex implies that $\bar{\Omega}$ is holomorphically convex in \mathbb{X} .

We showed that, for every point $p \in M_0$, the set \mathbb{E}_p of $f \in \mathcal{O}(\bar{\Omega})$ for which $f|_{M \cap \Omega}$ weakly CR-extends beyond p is of the first Baire category in $\mathcal{O}(\bar{\Omega})$. The set M_0 is separable. Then we take a dense sequence $\{p_\nu\}$ in M_0 and we observe that $\mathbb{E} = \bigcup_{\nu \in \mathbb{N}} \mathbb{E}_{p_\nu}$ is of the first Baire category, being a countable union of sets of the first Baire category. \square

6.4. The Cauchy problem for $\bar{\partial}_M$. Let $F^{\text{closed}}, U^{\text{open}} \subset M$. Set, for all $a = 0, \dots, \nu$, $q = 0, \dots, n$,

$$\mathcal{Q}^{a,q}(U; F) = \{f \in \mathcal{Q}^{a,q}(U) \mid f = 0^\infty \text{ on } F \cap U\}.$$

Clearly $(\mathcal{Q}^{a,*}(U; F), \bar{\partial}_M)$ is a subcomplex of $(\mathcal{Q}^{a,*}(U), \bar{\partial}_M)$ and we can consider the cohomology groups

$$H^{a,q}(U, F) = \ker(\bar{\partial}_M : \mathcal{Q}^{a,q}(U; F) \rightarrow \mathcal{Q}^{a,q+1}(U; F)) / \bar{\partial}_M \mathcal{Q}^{a,q-1}(U; F)$$

and also their germs

$$H^{a,q}(p_0, F) = \lim_{\longrightarrow p_0 \in U^{\text{open}}} H^{a,q}(U, F).$$

We obtain from Proposition 6.6

Proposition 6.7. *Let M be a CR submanifold of positive CR dimension n of a Stein manifold \mathbb{X} . Let M' be the set of points of M where M is minimal. Let Ω be a strictly pseudoconvex open subset of \mathbb{X} , with $M_0 = M' \cap \partial\Omega \neq \emptyset$. Then the set of $\alpha \in H^{0,1}(M, \bar{\Omega} \cap M)$ that restrict to the zero class of $H^{0,1}(p, \bar{\Omega} \cap M)$ for some $p \in M_0$ is infinite dimensional.*

More precisely, if we consider the Fréchet space

$$\mathcal{F} = \{f \in \Omega^1(M) \mid f|_{\bar{\Omega} \cap M} = 0^\infty, \quad df \in \mathcal{I}_M\},$$

then the set of $f \in \mathcal{F}$ for which we can find $p \in M_0$ and $u_{(p)} \in C_{(p)}^0$ with $u_{(p)}$ vanishing on $\bar{\Omega} \cap M$, and $\bar{\partial}_M u_{(p)} = f_{(p)}$, span a linear subspace of infinite dimensional codimension in \mathcal{F} . Here we wrote $f_{(p)}$ for the germ of f at p , and the equality $\bar{\partial}_M u_{(p)} = f_{(p)}$ has to be interpreted in the weak sense: it means that there is an open neighborhood U_p of p and a continuous function $u \in C(U_p)$, vanishing on $\bar{\Omega} \cap U_p$, such that

$$\int_M u d\eta = - \int_M f \wedge \eta, \quad \forall \eta \in (\mathcal{I}_M)^\vee(U_p) \cap \Omega_0^{2n+k-1}(U_p).$$

Proof. Let $g \in \mathcal{O}(\bar{\Omega})$. By Whitney's extension theorem, there is a smooth complex valued function \tilde{g} on \mathbb{X} with $\tilde{g} = g$ on $\bar{\Omega}$. Then $f = \bar{\partial}\tilde{g}|_M$ is an element of \mathcal{F} . Let $w = \tilde{g}|_M$. Then w is a smooth function on M and its restriction to $M \cap \Omega$ is CR. If $p \in M_0$ and $u_{(p)} \in C_{(p)}^0$ solves $\bar{\partial}_M u_{(p)} = f_{(p)}$, then $w_{(p)} + u_{(p)}$ yields a weak CR extension of $w|_{M \cap \Omega}$ beyond p . Therefore the thesis follows by Proposition 6.6. \square

Let N be a smooth submanifold of M . Its conormal bundle in M at $p \in N$ is

$$T_{N,p}^* M = \{\xi \in T_p^* M \mid \xi(v) = 0, \quad \forall v \in T_p N\}.$$

Definition 6.8. We say that N is *characteristic* at $p \in N$ if $T_{N,p}^* M \cap H_p^0 M \neq \{0\}$.

Equivalently, N is non characteristic at $p \in N$ if it contains a germ (N', p) of CR submanifold of type $(0, \nu)$ of M . We have the following

Lemma 6.9. *Let M be a CR manifold of positive CR dimension n , and N a smooth submanifold of M . Let U be an open neighborhood of a non characteristic point p_0 of N .*

If $f \in \mathcal{Q}^{0,1}(U, N)$ and $u \in C^\infty(U)$ solves

$$(6.11) \quad \begin{cases} \bar{\partial}_M u = f & \text{on } U, \\ u = 0 & \text{on } N \end{cases}$$

then u vanishes to infinite order at p_0 .

Proof. The statement follows by the uniqueness in the formal non characteristic Cauchy problem. \square

6.5. The canonical bundle. The sheaf $\mathcal{Q}^{v,0} = (\mathcal{J}_M)^v \cap \Omega_M^v$ is the sheaf of germs of smooth sections of a line bundle KM on M , that is called the *canonical bundle* of M .

Let $U^{\text{open}} \subset M$, and assume that $\theta_1, \dots, \theta_v \in \mathcal{J}_M(U)$ give a basis of $\mathbf{Z}_p^0 M$ at all points $p \in U$. Since $d\theta_j \in \mathcal{J}_M(U)$, we obtain

$$d(\theta_1 \wedge \dots \wedge \theta_v) = \alpha \wedge \theta_1 \wedge \dots \wedge \theta_v, \quad \text{with } \alpha \in \Omega^1(U).$$

The form α is uniquely determined modulo \mathcal{J}_M , and thus defines a unique element $[\alpha] \in \mathcal{Q}^{0,1}(U)$. By differentiating we get

$$\begin{aligned} 0 &= d^2(\theta_1 \wedge \dots \wedge \theta_v) = (d\alpha) \wedge \theta_1 \wedge \dots \wedge \theta_v - \alpha \wedge d(\theta_1 \wedge \dots \wedge \theta_v) \\ &= (d\alpha) \wedge \theta_1 \wedge \dots \wedge \theta_v - \alpha \wedge \alpha \wedge \theta_1 \wedge \dots \wedge \theta_v \\ &= (d\alpha) \wedge \theta_1 \wedge \dots \wedge \theta_v \iff \bar{\partial}_M[\alpha] = 0. \end{aligned}$$

If ω is another non zero section of KM , defined in a neighborhood of a point $p_0 \in U$, we have $\omega = e^u \theta_1 \wedge \dots \wedge \theta_v$ on a neighborhood U_{p_0} of p_0 in U and hence

$$d\omega = e^u(du + \alpha) \wedge \theta_1 \wedge \dots \wedge \theta_v \quad \text{on } U_{p_0}.$$

Thus we obtain

Lemma 6.10. *There is a section $\psi \in \Gamma(M, H^{0,1})$ of the sheaf of germs of cohomology classes of bidegree $(0, 1)$ such that*

$$(6.12) \quad \begin{cases} \forall p \in M, \quad \forall \omega \in \Gamma_{(p)}(KM), \quad \text{with } \omega(p) \neq 0, \quad \exists \alpha \in \Omega_{(p)}^1 \quad \text{such that} \\ d\omega(p) = \alpha(p) \wedge \omega(p), \quad \bar{\partial}_M[\alpha] = 0, \quad \llbracket \alpha \rrbracket \in \psi(p). \end{cases}$$

Here $\llbracket \alpha \rrbracket$ is the element of $H^{0,1}(p)$ defined by $[\alpha] \in \mathcal{Z}_{(p)}^{0,1}$.

Lemma 6.11. *If M is locally CR-embeddable at p , then $\psi(p) = 0$.*

If \mathcal{Z} is completely integrable at p , then, for $\alpha \in \Omega_{(p)}^1$ satisfying (6.12) there is $u \in C_{(p)}^0$ such that $\bar{\partial}_M u = [\alpha]$.

6.6. CR-foldings. Let us recall the notion of a fold singularity for a smooth map (see e.g. [13]). Let M, N be real smooth manifolds of the same real dimension. A map $\pi : M \rightarrow N$ is a *fold-map* if there is a smooth submanifold V of M such that:

- the restriction of π to $M \setminus V$ is a two-sheeted covering;
- the restriction of π to V is a smooth immersion;
- there is an involution $\sigma : U \rightarrow U$ of a tubular neighborhood U of V in M such that $\pi \circ \sigma = \pi$ on U .

The corresponding notion in CR geometry will be a folding about a CR-divisor. Let us introduce the notions that we will utilize in the sequel.

Definition 6.12 (Smooth CR-divisors). A *smooth CR-divisor* of a CR manifold M is a smooth submanifold D of M , having real codimension 2, such that for each $p_0 \in D$ there is an open neighborhood U_{p_0} of p_0 in M and a function $f \in \mathcal{O}_M^\infty(U_{p_0})$ such that $D \cap U_{p_0} = \{p \in U_{p_0} \mid f(p) = 0\}$ and $df(p_0) \wedge d\bar{f}(p_0) \neq 0$.

Definition 6.13 (CR-folding). Let M, N be CR manifolds having the same CR dimension. A *CR-folding* is a proper smooth CR map $\pi : M \rightarrow N$ such that there exists a smooth CR-divisor D on M with the properties:

- the restriction of π to $M \setminus D$ is a CR -submersion and a smooth circle bundle;
- the restriction of π to D is a smooth CR immersion;
- there is a tubular neighborhood U of D in M and an \mathbf{S}^1 -action on the fibers of $\pi|_U : U \rightarrow \pi(U)$, for which D is the set of fixed points.

Example 6.14. The map $S^3 = \{(z, w) \mid z\bar{z} + w\bar{w} = 1\} \ni (z, w) \xrightarrow{\pi} z \in \mathbb{C}$ is a CR -folding. The set $D = \{(z, 0) \mid z\bar{z} = 1\} = \{(z, w) \in S^3 \mid w = 0\}$ is a smooth CR -divisor in S^3 , and $\pi^{-1}(z_0) = \{(z_0, w) \mid w\bar{w} = 1 - z_0\bar{z}_0\}$. We can define, in the tubular neighborhood $U = S^3 \cap \{|z| > 0\}$, an \mathbf{S}^1 -action by $e^{i\theta} \cdot (z, w) = (z, e^{i\theta}w)$.

Example 6.15. Let $Q \subset \mathbb{CP}^v$, with $v \geq 3$, be the ruled real projective quadric, which is a CR submanifold of type $(n, 1)$, with $n = v - 1$, and Levi signature $(1, n - 1)$. For a suitable choice of homogeneous coordinates, Q has equation

$$(6.13) \quad z_0\bar{z}_0 + z_1\bar{z}_1 = \sum_{j=2}^v z_j\bar{z}_j.$$

The point $p_0 \equiv (1, 0, \dots, 0)$ does not belong to Q . Hence, by associating to each $p \in Q$ the complex line p_0p , we obtain a map $\pi : Q \rightarrow \mathbb{CP}^n$, where \mathbb{CP}^n is the set of complex lines through p_0 of \mathbb{CP}^{n+1} . After identifying \mathbb{CP}^n with the hyperplane $\{z_0 = 0\}$, the map π is described in homogeneous coordinates by

$$(6.14) \quad \pi : (z_0, z_1, \dots, z_v) \longrightarrow (z_1, \dots, z_v).$$

This map is a CR -folding of M into \mathbf{CP}^n , with divisor $Q \cap \{z_0 = 0\}$, whose image is the complement of an open ball in the projective space:

$$(6.15) \quad \pi(Q) = \{z_1\bar{z}_1 \leq \sum_{j=2}^v z_j\bar{z}_j\}.$$

To describe the generators of the ideal sheaf \mathcal{I}_Q of Q , we consider the covering $\{U, V\}$ of Q with $U = \{z_0 \neq 0\}$, $V = \{z_1 \neq 0\}$. Then \mathcal{I}_Q is defined by the generators

$$\begin{aligned} z_0^{-1}dz_1, z_0^{-1}dz_2, \dots, z_0^{-1}dz_v & \text{ on } U \cap Q, \\ z_1^{-1}dz_0, z_1^{-1}dz_2, \dots, z_1^{-1}dz_v & \text{ on } V \cap Q. \end{aligned}$$

The canonical bundle is generated by $z_0^{-v}dz_1 \wedge dz_2 \wedge \dots \wedge dz_v$ on $U \cap Q$ and by $z_1^{-v}dz_0 \wedge dz_2 \wedge \dots \wedge dz_v$ on $V \cap Q$.

6.7. Construction of a locally non CR -embeddable perturbation. In this subsection we describe a procedure to define a non locally CR -embeddable CR structure on a neighborhood of p_0 in M that we will use in §6.8 to produce a *global* example.

Let M, N be CR manifolds of type $(n, k), (n, k - 1)$ respectively. Assume that there is a CR map $\pi : M \rightarrow N$ having a Lorentzian singularity and a CR -folding at p_0 , and that M, N are locally CR -embeddable at p_0, q_0 , respectively. We are in fact in the situation of Lemma 5.4 and we keep the notation therein.

Let α be a $(0, 1)$ -form on ω , with $\bar{\partial}_N \alpha = 0$ and $\alpha = 0$ on ω_+ . Then $z_1^{-1}\pi^*\alpha$ is well defined and smooth because $\pi^*\alpha$ vanishes to infinite order on $D = \{z_1 = 0\}$, and we may consider on U the CR -structure with ideal sheaf \mathcal{I}'_U generated by

$$(6.16) \quad dz_1 - z_1^{-1}\pi^*\alpha, dz_2, \dots, dz_v.$$

This new CR structure agrees with the original one to infinite order at all points of D . If this new CR structure admits a CR -chart centered at p_0 , then there is a

smooth function u , defined on a neighborhood of p_0 , with $u(p_0) = 0$ and

$$(6.17) \quad d(e''(dz_1 - z_1^{-1}\pi^*\alpha) \wedge dz_2 \wedge \cdots \wedge dz_v) = 0.$$

Then we obtain

$$du \wedge (dz_1 - z_1^{-1}\pi^*\alpha) \wedge dz_2 \wedge \cdots \wedge dz_v + z_1^{-2}dz_1 \wedge \pi^*\alpha \wedge dz_2 \wedge \cdots \wedge dz_v = 0,$$

from which we get

$$(d(z_1^2 u) - \pi^*\alpha^*) \wedge \frac{dz_1}{z_1} \wedge dz_2 \wedge \cdots \wedge dz_v + d(u\pi^*\omega^* \wedge dz_2 \wedge \cdots \wedge dz_v) = 0.$$

Next we integrate on the fiber. For $q \in W \cap \omega_-$, where W is a suitable small neighborhood of q_0 in ω , we obtain

$$w(q) = \pi_*(z_1^2 u)(q) = \frac{1}{2\pi i} \oint_{\pi^{-1}(q)} z_1 u dz_1,$$

and therefore

$$\begin{aligned} dw(q) \wedge dz_2 \wedge \cdots \wedge dz_v &= \frac{1}{2\pi i} \oint_{\pi^{-1}(q)} d(z_1^2 u) \frac{dz_1}{z_1} \wedge dz_2 \wedge \cdots \wedge dz_v \\ &= \frac{1}{2\pi i} \oint_{\pi^{-1}(q)} \pi^*\alpha^* \frac{dz_1}{z_1} \wedge dz_2 \wedge \cdots \wedge dz_v = \alpha \wedge dz_2 \wedge \cdots \wedge dz_v, \end{aligned}$$

because

$$\oint_{\pi^{-1}(q)} d(u\pi^*\omega^* \wedge dz_2 \wedge \cdots \wedge dz_v) = 0.$$

We observe that $w = 0^2$ on $W \cap \partial\omega_-$ and that the equality established above means that w satisfies

$$\bar{\partial}_N w = [\alpha] \quad \text{on } N.$$

By Lemma 6.9, we actually have $w = 0^\infty$ on ∂N . By Proposition 6.7 there are $\bar{\partial}_N$ -closed forms α , of type $(0, 1)$, vanishing to infinite order on $W \cap \partial\omega_-$, for which the Cauchy problem

$$\begin{cases} \bar{\partial}_N w = \alpha & \text{on } W \cap \omega_-, \\ w = 0^\infty & \text{on } W \cap \partial\omega_- \end{cases}$$

has no solution when W is any open neighborhood of q_0 , hence yielding non CR -embeddable CR structures on U which agree to infinite order with the original one.

6.8. A global example. In this section we construct a non locally CR -embeddable CR structure on the Lorentzian quadric of Example 6.15.

In \mathbb{CP}^v , with homogeneous coordinates z_0, z_1, \dots, z_v , for $v \geq 2$, we consider the quadric $Q = \{z_0\bar{z}_0 + z_1\bar{z}_1 = z_2\bar{z}_2 + \cdots + z_v\bar{z}_v\}$. Fix the divisor $D = \{z_0 = 0\}$ and the global CR -folding $Q \rightarrow N$ where $N = \{z_1\bar{z}_1 \leq z_2\bar{z}_2 + \cdots + z_v\bar{z}_v\} \subset \mathbb{CP}^{v-1}$ is a strictly pseudoconcave closed domain in \mathbb{CP}^{v-1} . The closure of its complement is an Euclidean ball B of $\mathbb{C}^{v-1} = \mathbb{CP}^{v-1} \setminus \{z_1 = 0\}$. Fix a smooth function f , with compact support in \mathbb{C}^{v-1} , which is holomorphic on B , but cannot be continued holomorphically beyond any point of ∂B , and let α be the restriction of $\bar{\partial}f$ to N . We observe that $\zeta = \frac{z_1}{z_0}$ is meromorphic on \mathbf{CP}^m and that $\zeta\pi^*\alpha$ is well-defined on Q . We define a new CR structure on Q by the ideal sheaf having generators

$$\begin{aligned} z_1^{-1}dz_0 - \zeta\pi^*\alpha^*, z_1^{-1}dz_2, \dots, z_1^{-1}dz_v & \quad \text{on } M \cap \{z_1 \neq 0\}, \\ z_0^{-1}dz_1, z_0^{-1}dz_2, \dots, z_0^{-1}dz_v & \quad \text{on } M \setminus \text{supp } \pi^*\alpha. \end{aligned}$$

Note that the hyperplane $\{z_1 = 0\}$ in \mathbb{CP}^{v-1} does not intersect the support of α , and hence the ideal sheaf is well defined. By the argument in §6.7, with this new CR structure Q is not locally CR -embeddable at all points of the divisor D . Thus we have obtained

Theorem 6.16. *There are CR structures of type $(m-1, 1)$ on the Lorentzian quadric Q that are not locally CR -embeddable at all points of a hyperplane section of Q .* \square

REFERENCES

1. Takao Akahori, *A new approach to the local embedding theorem of CR -structures for $n \geq 4$ (the local solvability for the operator $\bar{\partial}_b$ in the abstract sense)*, Mem. Amer. Math. Soc. **67** (1987), no. 366, xvi+257.
2. Andrea Altomani, C. Denson Hill, Mauro Nacinovich, and Egmont Porten, *Complex vector fields and hypoelliptic partial differential operators*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 3, 987–1034.
3. Aldo Andreotti, Gregory Fredricks, and Mauro Nacinovich, *On the absence of Poincaré lemma in tangential Cauchy-Riemann complexes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 3, 365–404.
4. Aldo Andreotti and C. Denson Hill, *E. E. Levi convexity and the Hans Lewy problem. II. Vanishing theorems*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 747–806.
5. M. S. Baouendi and L. P. Rothschild, *Embeddability of abstract CR structures and integrability of related systems*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 3, 131–141.
6. M. S. Baouendi and Linda Preiss Rothschild, *Cauchy-Riemann functions on manifolds of higher codimension in complex space*, Invent. Math. **101** (1990), no. 1, 45–56.
7. M. S. Baouendi and F. Trèves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. (2) **113** (1981), no. 2, 387–421.
8. Shiferaw Berhanu, Paulo D. Cordaro, and Jorge Hounie, *An introduction to involutive structures*, New Mathematical Monographs, vol. 6, Cambridge University Press, Cambridge, 2008.
9. L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles*, Séminaire Goulaouic-Lions-Schwartz 1974–1975; Équations aux dérivées partielles linéaires et non linéaires, Centre Math., École Polytech., Paris, 1975, pp. Exp. No. 9, 14.
10. J. Brinkschulte, C. Denson Hill, and M. Nacinovich, *Obstructions to generic embeddings*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 6, 1785–1792.
11. Judith Brinkschulte, C. Denson Hill, and Mauro Nacinovich, *The Poincaré lemma and local embeddability*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **6** (2003), no. 2, 393–398.
12. David Catlin, *Sufficient conditions for the extension of CR structures*, J. Geom. Anal. **4** (1994), no. 4, 467–538.
13. Ja. M. Èliašberg, *Singularities of folding type*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 1110–1126.
14. C. Denson Hill, *What is the notion of a complex manifold with a smooth boundary?*, Algebraic analysis, Vol. I, Academic Press, Boston, MA, 1988, pp. 185–201.
15. ———, *Counterexamples to Newlander-Nirenberg up to the boundary*, Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 191–197.
16. C. Denson Hill and Mauro Nacinovich, *Embeddable CR manifolds with nonembeddable smooth boundary*, Boll. Un. Mat. Ital. A (7) **7** (1993), no. 3, 387–395.
17. ———, *Solvable Lie algebras and the embedding of CR manifolds*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **2** (1999), no. 1, 121–126.
18. ———, *On the failure of the Poincaré lemma for $\bar{\partial}_M$. II*, Math. Ann. **335** (2006), no. 1, 193–219.
19. Lars Hörmander, *On existence of solutions of partial differential equations*, Partial differential equations and continuum mechanics, Univ. of Wisconsin Press, Madison, Wis., 1961, pp. 233–240.

20. Howard Jacobowitz, *Simple examples of nonrealizable CR hypersurfaces*, Proc. Amer. Math. Soc. **98** (1986), no. 3, 467–468.
21. ———, *Homogeneous solvability and CR structures*, Notas de Curso [Course Notes], vol. 25, Universidade Federal de Pernambuco Departamento de Matemática, Recife, 1988.
22. Howard Jacobowitz and François Trèves, *Nonrealizable CR structures*, Invent. Math. **66** (1982), no. 2, 231–249.
23. ———, *Nowhere solvable homogeneous partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 3, 467–469.
24. Walter Koppelman, *The Cauchy integral for functions of several complex variables*, Bull. Amer. Math. Soc. **73** (1967), 373–377.
25. Masatake Kuranishi, *Strongly pseudoconvex CR structures over small balls. I. An a priori estimate*, Ann. of Math. (2) **115** (1982), no. 3, 451–500.
26. ———, *Strongly pseudoconvex CR structures over small balls. II. A regularity theorem*, Ann. of Math. (2) **116** (1982), no. 1, 1–64.
27. ———, *Strongly pseudoconvex CR structures over small balls. III. An embedding theorem*, Ann. of Math. (2) **116** (1982), no. 2, 249–330.
28. Hans Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. (2) **64** (1956), 514–522.
29. ———, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158.
30. Lan Ma and Joachim Michel, *Regularity of local embeddings of strictly pseudoconvex CR structures*, J. Reine Angew. Math. **447** (1994), 147–164.
31. Abdelhamid Meziani, *Perturbation of a class of CR structures of codimension larger than one*, J. Funct. Anal. **116** (1993), no. 1, 225–244.
32. M. Nacinovich, *Poincaré lemma for tangential Cauchy-Riemann complexes*, Math. Ann. **268** (1984), no. 4, 449–471.
33. M. Nacinovich and E. Porten, *C^∞ -hypoellipticity and extension of cr functions*, arXiv:1107.3374 (2011).
34. A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.
35. Louis Nirenberg, *A certain problem of Hans Lewy*, Uspehi Mat. Nauk **29** (1974), no. 2(176), 241–251, Translated from the English by Ju. V. Egorov, Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), I.
36. Louis Nirenberg and François Trèves, *On local solvability of linear partial differential equations. I. Necessary conditions*, Comm. Pure Appl. Math. **23** (1970), 1–38.
37. ———, *On local solvability of linear partial differential equations. II. Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–509.
38. S. I. Pinchuk and S. V. Khasanov, *Asymptotically holomorphic functions and their applications*, Mat. Sb. (N.S.) **134(176)** (1987), no. 4, 546–555, 576.
39. H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, second ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999.
40. F. Trèves, *Approximation and representation of functions and distributions annihilated by a system of complex vector fields*, École Polytechnique Centre de Mathématiques, Palaiseau, 1981.
41. François Trèves, *Hypo-analytic structures*, Princeton Mathematical Series, vol. 40, Princeton University Press, Princeton, NJ, 1992, Local theory.
42. A. E. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type*, Mat. Sb. (N.S.) **136(178)** (1988), no. 1, 128–139.
43. ———, *Extension of CR-functions into a wedge*, Mat. Sb. **181** (1990), no. 7, 951–964.

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